

INTEGRAL REPRESENTATIONS OF *-REPRESENTATIONS OF *-ALGEBRAS

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ABSTRACT. Regular normalized $B(W_1, W_2)$ -valued non-negative spectral measures introduced in [16] are in one-to-one correspondence with unital *-representations $\rho : C(X, \mathbb{C}) \otimes W_1 \rightarrow W_2$, where X stands for a compact Hausdorff space and W_1, W_2 stand for von Neumann algebras. In this paper we generalize this result in two directions. The first is to *-representations of the form $\rho : \mathcal{B} \otimes W_1 \rightarrow W_2$, where \mathcal{B} stands for a commutative *-algebra \mathcal{B} , and the second is to special (not necessarily bounded) *-representations of the form $\rho : \mathcal{B} \otimes W_1 \rightarrow \mathcal{L}^+(\mathcal{D})$, where $\mathcal{L}^+(\mathcal{D})$ stands for a *-algebra of special linear operators on a dense subspace \mathcal{D} of a Hilbert space \mathcal{K} .

1. INTRODUCTION AND NOTATION

A *-representation of a C^* -algebra \mathcal{A} is an algebra homomorphism $\rho : \mathcal{A} \rightarrow W$ such that $\rho(a^*) = \rho(a)^*$, where W stands for a von Neumann algebra. A version of a well-known representation theorem is the following (see [5, p. 259] or [6, 5.2.6. Theorem] and note that the C^* -algebra $B(\mathcal{H})$ of bounded linear operators on a Hilbert space \mathcal{H} can be replaced by a von Neumann algebra W by [9, Theorem 2.7.4]).

Theorem 1.1. *Let X be a compact Hausdorff space, W a von Neumann algebra and $\rho : C(X, \mathbb{C}) \rightarrow W$ a linear map. Let $\text{Bor}(X)$ be a Borel σ -algebra on X . The following statements are equivalent.*

- (1) $\rho : C(X, \mathbb{C}) \rightarrow W$ is a unital *-representation.
- (2) There exists a unique regular normalized spectral measure $E : \text{Bor}(X) \rightarrow W$ such that $\rho(f) = \int_X f dE$ for every $f \in C(X, \mathbb{C})$.

The motivation for our previous paper [16] was a generalization of Theorem 1.1 to certain non-commutative C^* -algebras. Namely, we found a generalization to the *-representations of the form $\rho : C(X, W_1) \rightarrow W_2$, where W_1, W_2 stand for von Neumann algebras. For the generalization we introduced non-negative spectral measures. *Non-negative spectral measure* $M : \text{Bor}(X) \rightarrow B(W_1, W_2)$ is a set function, such that for every hermitian projection $P \in W_1$ the set functions $M_P : \text{Bor}(X) \rightarrow W_2, M_P(\Delta) := M(\Delta)(P)$ are spectral measures and the equality $M_P(\Delta_1)M_Q(\Delta_2) = M_{PQ}(\Delta_1 \cap \Delta_2)$ holds for all hermitian projections $P, Q \in W_1$ and all sets $\Delta_1, \Delta_2 \in \text{Bor}(X)$. The generalization is the following.

Theorem 1.2. *Let X be a compact Hausdorff space, $\text{Bor}(X)$ a Borel σ -algebra on X , W_1, W_2 von Neumann algebras and $\rho : C(X, W_1) \rightarrow W_2$ a map. The following statements are equivalent.*

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- (1) $\rho : C(X, W_1) \rightarrow W_2$ is a unital $*$ -representation.
- (2) There exists a unique regular normalized non-negative spectral measure $M : \text{Bor}(X) \rightarrow B(W_1, W_2)$ such that $\rho(F) = \int_X F dM$ for every $F \in C(X, W_1)$.

Since the common assumption in Theorems 1.1 and 1.2 is the boundedness of $*$ -representations, our initial motivation for research was to study unbounded versions of Theorems 1.1 and 1.2. Our initial setting were the algebras $C(X, \mathbb{C})$ and $C(X, \mathbb{C}) \otimes W$, where X is not a compact Hausdorff space. However, in the course of research, the natural question of replacing the algebra $C(X, \mathbb{C})$ by a general $*$ -algebra \mathcal{B} appeared. For this purpose, Theorems 1.1 and 1.2 have to be generalized to general $*$ -algebras first. The motivation for this paper is the following problem.

Problem. *Let \mathcal{B} be a commutative unital $*$ -algebra and W, W_1, W_2 von Neumann algebras. Find one-to-one correspondence between $*$ -representations ρ of*

- (A) \mathcal{B} , which map into a von Neumann algebra W , and special spectral measures.
- (B) $\mathcal{B} \otimes W_1$, which map into a von Neumann algebra W_2 , and special non-negative spectral measures.
- (C) \mathcal{B} , which map into a $*$ -algebra of all linear operators on a densely defined subspace of a Hilbert space, and special spectral measures.
- (D) $\mathcal{B} \otimes W$, which map into a $*$ -algebra of all linear operators on a densely defined subspace of a Hilbert space, and special non-negative spectral measures.

Before introducing terminology about $*$ -algebra and their $*$ -representations, let us briefly survey some known facts about integral representations from literature.

Integral representations of commutative semigroups with involution in literature. A usual and even more general context, is the study of integral representations (via spectral measures) of $*$ -representations of commutative semigroups S with involution $*$ on a Hilbert space (where the operators are not necessarily bounded). The integration is over the set S^* of all characters of S , i.e., functions $\chi : S \rightarrow \mathbb{C}$ such that $\chi(st^*) = \chi(s)\chi(t)^*$ holds for every $s, t \in S$. S^* is a completely regular space when equipped with the topology of pointwise convergence inherited from \mathbb{C}^S . The cases $S = (-\infty, \infty)$ or $S = [0, \infty)$ with multiplication as the semigroup operation are well-known, see [8, Chapter XI]. The extension to general commutative semigroups with identity and involution are [10, Theorem 2] and [1, Theorem 3.1]. [10, Theorem 2] is equivalent to Theorem 1.1 above. The main step in the proof of the equivalence is to show, that if a semigroup with an involution is induced by a unital, commutative complex algebra with an involution, where the semigroup operation is the algebra multiplication, then the spectral measure E is supported on the subspace of linear characters. The extension to unbounded operators is also deeply studied, see [11] and reference therein. The most general representation theorem (with unique regular spectral measure), where there are no topological constraints on the semigroup S involved, is [11, Theorem 1.2]. The proof is very technical but the main step is the reduction to the bounded case [10, Theorem 2] and then the constructed spectral measure on a certain dense subspace (D_c in the notation of [11]) is extended by continuity to the whole Hilbert space. The density of D_c is included in the definition of $*$ -representation from [11] (see [11, Definition 1.1]). Since D_c is always dense if we integrate with respect to a regular spectral measure, this is a natural requirement.

Terminology on *-algebras and on *-representations. We follow the monograph [14]. Let \mathcal{D} be a dense linear subspace in a Hilbert space \mathcal{H} .

A **-algebra* is a complex associative algebra \mathcal{A} equipped with a mapping $a \rightarrow a^*$, called the *involution* of \mathcal{A} , such that $(\lambda a + \mu b)^* = \bar{\lambda}a^* + \bar{\mu}b^*$ and $(a^*)^* = a$ for all $a, b \in \mathcal{A}$ and all $\lambda, \mu \in \mathbb{C}$. $\mathcal{L}^+(\mathcal{D})$ is the set of all linear operators a with domain \mathcal{D} for which $a\mathcal{D} \subseteq \mathcal{D}$, $\mathcal{D} \subseteq \mathcal{D}(a^*)$ and $a^*\mathcal{D} \subseteq \mathcal{D}$, where a^* stands for the classical adjoint of an operator a . $\mathcal{L}^+(\mathcal{D})$ equipped with an involution $a^+ = a^* \upharpoonright_{\mathcal{D}}$ is a **-algebra*. Note that the identity operator $\text{Id}_{\mathcal{D}}$ belongs to $\mathcal{L}^+(\mathcal{D})$ and by [14, Proposition 2.1.8], all $a \in \mathcal{L}^+(\mathcal{D})$ are closable. We write \bar{a} for the *closure* of an operator a .

A **-representation* of a **-algebra* \mathcal{A} on \mathcal{D} is an algebra homomorphism ρ of \mathcal{A} into $\mathcal{L}^+(\mathcal{D})$ such that $\rho(1) = \text{Id}_{\mathcal{D}}$ and $\langle \rho(a)\phi, \psi \rangle = \langle \phi, \rho(a^*)\psi \rangle$ for all $\phi, \psi \in \mathcal{D}$ and all $a \in \mathcal{A}$. The *graph topology* of ρ is the locally convex topology on the vector space \mathcal{D} defined by the norms $h \rightarrow \|\phi\| + \|\rho(a)\phi\|$, where $a \in \mathcal{A}$. If $\mathcal{D}(\bar{\rho})$ denotes the completion of \mathcal{D} in the graph topology of ρ , then $\bar{\rho}(a) := \rho(a) \upharpoonright_{\mathcal{D}(\bar{\rho})}$, $a \in \mathcal{A}$, defines a **-representation* of \mathcal{A} with domain $\mathcal{D}(\bar{\rho})$, called the *closure* of ρ . In particular, ρ is *closed* if and only if \mathcal{D} is complete in the graph topology of ρ . A closed **-representation* ρ of a commutative **-algebra* \mathcal{B} is called *integrable* if $\overline{\rho(b^*)} = \rho(b)^*$ for all $b \in \mathcal{B}$. For several characterizations of the integrability, see [14, Chapter 9].

Character space $\widehat{\mathcal{B}}$ of a commutative **-algebra* \mathcal{B} is the set of all non-trivial **-homomorphisms* $\chi : \mathcal{B} \rightarrow \mathbb{C}$. An element $b \in \mathcal{B}$ can be viewed as a function f_b on the set $\widehat{\mathcal{B}}$, i.e., $f_b(\chi) = \chi(b)$ for $b \in \mathcal{B}$ and $\chi \in \widehat{\mathcal{B}}$. Let τ denote the weakest topology on the set $\widehat{\mathcal{B}}$ for which all functions f_b are continuous. This topology is generated by the sets $f_b^{-1}((c, d))$, $-\infty \leq c \leq d \leq \infty$. The topology τ on $\widehat{\mathcal{B}}$ is Hausdorff.

Comparison of *-representations of semigroups with involution and *-algebras. A possible approach to solve Problems A-D is to induce a semigroup with involution from the given commutative **-algebra* and use the results on integral representations of **-representations* of semigroups with involution presented above. However, there is a problem, because of the differences in the definitions of **-representations* of a **-algebra* (see [14]) and a semigroup (see [10] and [11]). The differences are the following:

- (1) **-representations* of semigroups are not ‘linear’ compared to **-algebras*, where they are.
- (2) **-representation* of a semigroup maps into the set of normal operators (not necessarily having the same domain) while **-representation* of a **-algebra* maps into the set of all linear operators on a chosen dense subspace in a Hilbert space, obeying some additional conditions.
- (3) **-representation* of a semigroup has all the necessary properties to be an integration with respect to a regular spectral measure, while in the case of a **-algebra* this is not required.

Hence, there are more **-representations* of a **-algebra*. Therefore, one has to be cautious when using the results for semigroups in the context of **-algebras*. However, if we concentrate on **-representations* with a regular representing spectral measures, they can be used. In the next subsection we prove two properties of such **-representations*.

***-representations of commutative *-algebras with a regular representing spectral measures.** Let $\rho : \mathcal{B} \rightarrow \mathcal{L}^+(\mathcal{D})$ be a **-representation* of a commutative

*-algebra \mathcal{B} on a dense subspace \mathcal{D} of a Hilbert space \mathcal{H} . We say, ρ has an *integral representation* if there is a spectral measure $E : \text{Bor}(\widehat{\mathcal{B}}) \rightarrow B(\mathcal{H})$ on a Borel σ -algebra $\text{Bor}(\widehat{\mathcal{B}})$, such that for every $b \in \mathcal{B}$ we have $\overline{\rho(b)}x = (\int_{\widehat{\mathcal{B}}} f_b(\chi) dE(\chi))x$ for every $x \in \mathcal{D}(\overline{\rho(b)})$.

The following proposition states, that every *-representation $\rho : \mathcal{B} \rightarrow \mathcal{L}^+(\mathcal{D})$ with an integral representation is integrable.

Proposition 1.3. *Assume the notation above. If $\rho : \mathcal{B} \rightarrow \mathcal{L}^+(\mathcal{D})$ has an integral representation, then it is integrable.*

Proof. Since the spectral integral is always a normal operator (see (iv) of Theorem 2.6 below), $\overline{\rho(b)}$ must be normal for every $b \in \mathcal{B}$. By [14, Theorem 9.1.2], this is true iff ρ is integrable. \square

Now we derive a necessary condition for the regularity of the representing spectral measure $E : \text{Bor}(\widehat{\mathcal{B}}) \rightarrow B(\mathcal{H})$ of $\rho : \mathcal{B} \rightarrow \mathcal{L}^+(\mathcal{D})$. We say that E is *regular*, if for every $h_1, h_2 \in \mathcal{H}$ the complex measure $E_{h_1, h_2} : \text{Bor}(\widehat{\mathcal{B}}) \rightarrow \mathbb{C}$, defined by $E_{h_1, h_2}(\Delta) := \langle E(\Delta)h_1, h_2 \rangle$ is regular. Let \mathcal{K} be the set of compact sets in $\widehat{\mathcal{B}}$. For a compact set $K \subseteq \text{Bor}(\widehat{\mathcal{B}})$ we define the function $\alpha_K : C(\widehat{\mathcal{B}}, \mathbb{C}) \rightarrow \mathbb{R}$ by $\alpha(K)(f) := \sup_{\chi \in K} \|f(\chi)\|$, where $C(\widehat{\mathcal{B}}, \mathbb{C})$ is the vector space of continuous functions. Let $\rho : \mathcal{B} \rightarrow \mathcal{L}^+(\mathcal{D})$ be a *-representation and $K \in \mathcal{K}$ a compact set. Define the set

$$D_{\alpha_K, \rho} := \left\{ h \in \cap_{b \in \widehat{\mathcal{B}}} \mathcal{D}(\overline{\rho(b)}) : \|\overline{\rho(b)}h\| \leq \alpha_K(b)\|h\| \right\}.$$

The following proposition states, that if a representing measure E of ρ is regular, then $\cup_{K \in \mathcal{K}} D_{\alpha_K, \rho}$ is dense in \mathcal{H} .

Proposition 1.4. *Assume the notation above. If ρ has an integral representation by a regular spectral measure E , then the set $\cup_{K \in \mathcal{K}} D_{\alpha_K, \rho}$ is dense in \mathcal{H} .*

Proof. First we prove, that the set $\cup_{K \in \mathcal{K}} E(K)(\mathcal{H})$ is dense in \mathcal{H} . Pick $h \in \mathcal{H}$. For every $K \in \mathcal{K}$ it is true that

$$\begin{aligned} \|h - E(K)h\| &= \|E(K^c)h\| = (E_{h, h}(K^c))^{\frac{1}{2}} = (E_{h, h}(X) - E_{h, h}(K))^{\frac{1}{2}} \\ &= (\|h\|^2 - E_{h, h}(K))^{\frac{1}{2}}. \end{aligned}$$

By the regularity of $E_{h, h}$, we have $\sup_{K \in \mathcal{K}} E_{h, h}(K) = \|h\|^2$. Hence, we conclude that the set $\cup_{K \in \mathcal{K}} E(K)(\mathcal{H})$ is dense in \mathcal{H} .

Now we prove that the set $\cup_{K \in \mathcal{K}} D_{\alpha_K, \rho}$ contains the set $\cup_{K \in \mathcal{K}} E(K)(\mathcal{H})$. For every $h \in \cup_{K \in \mathcal{K}} E(K)(\mathcal{H})$ there is a compact set $K \in \mathcal{K}$ such that $h \in E(K)\mathcal{H}$. Then for every $b \in \mathcal{B}$ we have

$$\int_{\widehat{\mathcal{B}}} |f_b|^2 dE_{h, h} = \int_K |f_b|^2 dE_{h, h} \leq \alpha_K(b)^2 \int_K 1 dE_{h, h} \leq \alpha_K(b)^2 \|h\|^2.$$

Hence, $\cup_{K \in \mathcal{K}} E(K)(\mathcal{H}) \subseteq \cup_{K \in \mathcal{K}} D_{\alpha_K, \rho}$. Thus, $\cup_{K \in \mathcal{K}} D_{\alpha_K, \rho}$ is dense in \mathcal{H} . \square

By Propositions 1.3 and 1.4, *-representations $\rho : \mathcal{B} \rightarrow \mathcal{L}^+(\mathcal{D})$ with a representing spectral measure $E : \text{Bor}(\widehat{\mathcal{B}}) \rightarrow B(\mathcal{H})$ are integrable and have a dense set $\cup_{K \in \mathcal{K}} D_{\alpha_K, \rho}$.

Problem C - known result. For a special case of a countably generated commutative unital *-algebra \mathcal{B} , the solution to Problem C is the following result (see [13, Theorem 7]).

Theorem C. *Suppose that \mathcal{B} is a countably generated commutative unital *-algebra. Suppose $\widehat{\mathcal{B}}$ is equipped with the Borel structure induced by the weakest topology for which all functions f_b , $b \in \mathcal{B}$, are continuous. Let $\rho : \mathcal{B} \rightarrow \mathcal{L}^+(\mathcal{D})$ be a closed *-representation of \mathcal{B} on a dense linear subspace \mathcal{D} of a Hilbert space \mathcal{H} . The following statements are equivalent.*

- (1) ρ is an integrable *-representation of \mathcal{B} .
- (2) *There exists a unique spectral measure $E_\rho : \text{Bor}(\widehat{\mathcal{B}}) \rightarrow B(\mathcal{H})$ such that $\overline{\rho(b)} = \int_{\widehat{\mathcal{B}}} f_b(\chi) dE(\chi)$ for all $b \in \mathcal{B}$.*

Theorem C gives a one-to-one correspondence between integrable *-representations of a countably generated commutative unital *-algebra \mathcal{B} on a dense linear subspace \mathcal{D} of a Hilbert space \mathcal{H} and spectral measures E on $\text{Bor}(\widehat{\mathcal{B}})$ such that $(\int_{\widehat{\mathcal{B}}} f_b(\chi) dE(\chi))\mathcal{D} \subseteq \mathcal{D}$ for every $b \in \mathcal{B}$. However, Theorem C covers only countably generated commutative unital *-algebras \mathcal{B} , e.g., $\mathcal{B} := \mathbb{C}[x_1, \dots, x_n]$. The case $\mathcal{B} = C(X, \mathbb{C})$, where X is a topological space, is not covered.

Problem - new results. In what follows \mathcal{B} will be a commutative unital *-algebra, $\widehat{\mathcal{B}}$ the character space of \mathcal{B} equipped with the Borel structure $\text{Bor}(\widehat{\mathcal{B}})$ induced by the weakest topology for which all functions f_b , $b \in \mathcal{B}$, are continuous, \mathcal{H}, \mathcal{K} Hilbert spaces and $W \subseteq B(\mathcal{H}), W_1 \subseteq B(\mathcal{H})$ von Neumann algebras.

Recall, that the support of a spectral measure $E : \text{Bor}(\widehat{\mathcal{B}}) \rightarrow W$, denoted by $\text{supp}(E)$, is the set $\bigcup_{h \in \mathcal{H}} \text{supp}(E_{h,h})$ in $\widehat{\mathcal{B}}$. The following is the solution to Problem A.

Theorem A. *The following statements are equivalent.*

- (1) $\rho : \mathcal{B} \rightarrow W$ is a unital *-representation.
- (2) *There exists a unique regular spectral measure $E : \text{Bor}(\widehat{\mathcal{B}}) \rightarrow W$ with a compact support such that $\rho(b) = \int_{\widehat{\mathcal{B}}} f_b(\chi) dE(\chi)$ for all $b \in \mathcal{B}$.*

For every $F := \sum_{j=1}^m b_j \otimes A_j \in \mathcal{B} \otimes W_1$ we define a map $f_F(\chi) : \widehat{\mathcal{B}} \otimes W_1 \rightarrow W_1$ by $f_F(\chi) := \sum_{j=1}^m f_{b_j}(\chi) \otimes A_j$. The support of a non-negative spectral measure $M : \text{Bor}(\widehat{\mathcal{B}}) \rightarrow B(W_1, W_2)$, denoted by $\text{supp}(M)$, is the support of the spectral measure $M_{\text{Id}_{\mathcal{H}}} : \text{Bor}(\widehat{\mathcal{B}}) \rightarrow W_2, M_{\text{Id}_{\mathcal{H}}}(\Delta) := M(\Delta)(\text{Id}_{\mathcal{H}})$. Using Theorem A we obtain the following solution to Problem B.

Theorem B. *The following statements are equivalent.*

- (1) $\rho : \mathcal{B} \otimes W_1 \rightarrow W_2$ is a unital *-representation.
- (2) *There exists a unique regular normalized non-negative spectral measure $M : \text{Bor}(\widehat{\mathcal{B}}) \rightarrow B(W_1, W_2)$ with a compact support such that $\rho(F) = \int_{\widehat{\mathcal{B}}} f_F(\chi) dM(\chi)$ for every $F \in \mathcal{B} \otimes W_1$.*

Let us now study Problem C. If we concentrate only on regular spectral measures, then Propositions 1.3 in 1.4 give necessary conditions for ρ to have a representing measure. Theorem C' states, that they are also sufficient.

Theorem C'. *Let \mathcal{D} be a dense linear subspace \mathcal{D} of a Hilbert space \mathcal{H} and $\rho : \mathcal{B} \rightarrow \mathcal{L}^+(\mathcal{D})$ a *-representation. The following statements are equivalent.*

- (1) ρ is integrable and the set $\cup_{K \in \mathcal{K}} D_{\alpha_K, \rho}$ is dense in \mathcal{H} .
- (2) There exists a unique regular spectral measure $E : \text{Bor}(\widehat{\mathcal{B}}) \rightarrow B(\mathcal{H})$, such that $\text{supp}(E_{h,h})$ is compact for every $h \in \cup_{K \in \mathcal{K}} D_{\alpha_K, \rho}$ and the equality $\overline{\rho(b)}x = \int_{\widehat{\mathcal{B}}} f_b(\chi) dE(\chi) x$ holds for every $x \in \mathcal{D}(\overline{\rho(b)})$ and all $b \in \mathcal{B}$.

Let \mathcal{D} be a dense linear subspace of a Hilbert space \mathcal{K} and $\rho : \mathcal{B} \otimes W \rightarrow \mathcal{L}^+(\mathcal{D})$ a $*$ -representation. For every hermitian projection $P \in B(\mathcal{H})$ the map $\rho_P : \mathcal{B} \rightarrow \mathcal{L}^+(\mathcal{D})$, defined by $\rho_P(b) := \rho(b \otimes P)$, is a $*$ -representation. Using Theorem C' we obtain the following solution to Problem D.

Theorem D. *For a $*$ -representation $\rho : \mathcal{B} \otimes W \rightarrow \mathcal{L}^+(\mathcal{D})$ the following statements are equivalent.*

- (1) The map ρ_P is integrable $*$ -representation for every hermitian projection $P \in W$ and the set $\cup_{K \in \mathcal{K}} \mathcal{D}_{\alpha_K, \rho|_{\text{Id}_{\mathcal{H}}}}$ is dense in \mathcal{K} .
- (2) There exists a unique regular normalized non-negative spectral measure $M : \text{Bor}(\widehat{\mathcal{B}}) \rightarrow B(W, B(\mathcal{K}))$ such that $\overline{\rho(F)}x = \int_{\widehat{\mathcal{B}}} f_F(\chi) dM(\chi) x$ holds for every $x \in \mathcal{D}(\overline{\rho(F)})$ and all $F \in \mathcal{B} \otimes W$.

The paper is organized in the following way. In Section 2 we first introduce some theory needed throughout the paper, i.e., in Subsection 2.1 we recall a theory of non-negative spectral measures and integration with respect to them, in Subsection 2.2 we repeat the integration of unbounded functions with respect to the spectral measure and finally, and in Subsection 2.3, we present two results on the integral representation of $*$ -representations of a commutative semigroup with an involution. Theorems A and B are proved in Section 3, while Theorem C' in Section 4. Before proving Theorem D, the theory of integration with respect to the non-negative spectral measure has to be extended to the integration of unbounded functions. This is done in Section 5. Finally, in Section 6, Theorem D is proved.

2. PRELIMINARIES

In Subsection 2.1 we present the non-negative spectral measures, which we introduced in [16] to prove a theorem on the integral representation of the $*$ -representation of a C^* -algebra (see Theorem 1.2). In Subsection 2.2 we present a spectral theory of unbounded functions on a Hilbert space (see [2] or [15]). In Subsection 2.3 we recall results for integral representations of $*$ -representations of commutative semigroups with an involution from [10], [11].

2.1. Non-negative spectral measures. Let $(X, \text{Bor}(X), W_1, W_2)$ be a measure space, i.e., X is a topological space, $\text{Bor}(X)$ a σ -algebra on X , $W_1 \subseteq B(\mathcal{H})$, $W_2 \subseteq B(\mathcal{K})$ von Neumann algebras, where $B(\mathcal{H}), B(\mathcal{K})$ denote the bounded linear operators on Hilbert spaces \mathcal{H}, \mathcal{K} . We denote by W_p, W_+ the subsets of all hermitian projections and all positive operators of a von Neumann algebra W . By a hermitian projection we mean an operator P , which satisfies $P = P^* = P^2$ and by a positive operator we mean a hermitian operator A , such that $\langle Ah, h \rangle \geq 0$ for every $h \in \mathcal{H}$, where \mathcal{H} is a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, such that $W \subseteq B(\mathcal{H})$. Non-negative spectral measure $M : \text{Bor}(X) \rightarrow B(W_1, W_2)$ is a set function, if for every hermitian projection $P \in (W_1)_p$ the set functions $M_P : \text{Bor}(X) \rightarrow W_2$, defined by $M_P(\Delta) := M(\Delta)(P)$, are spectral measures and the equality $M_P(\Delta_1)M_Q(\Delta_2) = M_{PQ}(\Delta_1 \cap \Delta_2)$ holds for all hermitian projections

$P, Q \in (W_1)_p$ and all sets $\Delta_1, \Delta_2 \in \text{Bor}(X)$. Before writing a characterization of non-negative spectral measures, we need the following definition.

Definition 2.1. Let W be a von Neumann algebra and $A \in W$ a hermitian operator. Let $E : \text{Bor}([a, b]) \rightarrow W$ be its representing spectral measure, i.e., $A = \int_{[a, b]} \lambda dE(\lambda)$ with the spectrum $\sigma(A)$ of A lying in $[a, b]$. For $\ell \in \mathbb{N}$, let $\mathcal{Z}_\ell = \{\lambda_0, \lambda_1, \dots, \lambda_{n_\ell}\}$ be a partition of $[a - 1, b]$, such that $a - 1 < \lambda_{0, \ell} < a < \lambda_{1, \ell} < \dots < \lambda_{1, n_\ell} < b$ and $\mathcal{Z}_\ell \subset \mathcal{Z}_{\ell+1}$. Define a sequence $S_\ell(A)$ of Riemann sums of the form

$$S_\ell(A) = \sum_{k=1}^{n_\ell} \zeta_{k, \ell} (E(\lambda_{k, \ell}) - E(\lambda_{k-1, \ell})) =: \sum_{k=1}^{n_\ell} \zeta_{k, \ell} R_{k, \ell},$$

where the family $\{E(\lambda) \mid \lambda \in \mathbb{R}\}$ is the resolution of identity corresponding to the spectral measure E and $\zeta_{k, \ell} \in [\lambda_{k-1, \ell}, \lambda_{k, \ell}]$. Let $|\mathcal{Z}_\ell| := \max_k |\lambda_{k, \ell} - \lambda_{k-1, \ell}|$. We can choose $|\mathcal{Z}_\ell|$ small enough, so that $\|A - S_\ell(A)\| \leq \frac{1}{\ell}$. A sequence $S_\ell(A)$ is called a *limiting sequence* of A .

A characterization of non-negative spectral measures [16, Theorem 8.1] is the following.

Theorem 2.2. Let $(X, \text{Bor}(X), W_1, W_2)$ be a measure space and $\{E_P\}_{P \in (W_1)_p}$ a family of spectral measures $E_P : \text{Bor}(X) \rightarrow W_2$.

There is a unique non-negative spectral measure M such that

$$M_P = E_P$$

for all hermitian projections $P \in (W_1)_p$ iff the following conditions hold.

$$(1) \quad \sum_{i=1}^n \lambda_i E_{P_i}(\Delta) = \sum_{j=1}^m \mu_j E_{Q_j}(\Delta)$$

for all hermitian projections $P_i, Q_j \in (W_1)_p$, all real numbers $\lambda_i, \mu_j \in \mathbb{R}$, and all sets $\Delta \in \text{Bor}(X)$ such that $\sum_{i=1}^n \lambda_i P_i = \sum_{j=1}^m \mu_j Q_j$, for each set $\Delta \in \text{Bor}(X)$ there exists a constant $k_\Delta \in \mathbb{R}^{>0}$ such that

$$(2) \quad \|E_P(\Delta)\| \leq k_\Delta$$

for all hermitian projections $P \in (W_1)_p$, and for all hermitian projections $P, Q \in (W_1)_p$ and all sets $\Delta_1, \Delta_2 \in \text{Bor}(X)$ the equality

$$(3) \quad E_P(\Delta_1)E_Q(\Delta_2) = \lim_{\ell \rightarrow \infty} \sum_{j=0}^3 i^j \sum_{k=1}^{n_\ell} \zeta_{k, \ell, j} E_{R_{k, \ell, j}}(\Delta_1 \cap \Delta_2)$$

holds for limiting sequences $S_{\ell, j} := \sum_{k=1}^{n_\ell} \zeta_{k, \ell, j} R_{k, \ell, j}$ of the operators $\text{Re}(PQ)_+$, $\text{Im}(PQ)_+$, $\text{Re}(PQ)_-$, $\text{Im}(PQ)_-$ for $j = 0, 1, 2, 3$ respectively, where the decomposition of PQ into the linear combination of positive parts is $PQ = \text{Re}(PQ)_+ - \text{Re}(PQ)_- + i \cdot \text{Im}(PQ)_+ - i \cdot \text{Im}(PQ)_-$.

From the proof of [16, Theorem 8.1], we extract the following proposition, which will be needed in this paper.

Proposition 2.3. *Suppose $\{E_P\}_{P \in (W_1)_p}$ is a family of spectral measures $E_P : \text{Bor}(X) \rightarrow W_2$ satisfying conditions (1) and (2) of Theorem 2.2. For any positive operator $A \in W_1$ with a limiting sequence $S_\ell(A)$, the measure*

$$E_A : \text{Bor}(X) \rightarrow W_2, \quad E_A(\Delta) := \lim_{\ell \rightarrow \infty} E_{S_\ell(A)}(\Delta),$$

is a positive operator-valued measure. Moreover, the definition of E_A is independent of the choice of a limiting sequence $S_\ell(A)$.

Let X be a topological space and $\text{Bor}(X)$ a Borel σ -algebra on X . Non-negative spectral measure M is *regular* if the spectral measures M_P are regular for every hermitian projection $P \in (W_1)_p$, i.e., complex measures $(M_P)_{k_1, k_2} : \text{Bor}(X) \rightarrow \mathbb{C}$, $(M_P)_{k_1, k_2}(\Delta) := \langle M_P(\Delta)k_1, k_2 \rangle$ are regular for every $k_1, k_2 \in \mathcal{K}$ and every $P \in (W_1)_p$. M is *normalized* if $M(X)(\text{Id}_{\mathcal{H}}) = \text{Id}_{\mathcal{K}}$, where $\text{Id}_{\mathcal{H}}$, $\text{Id}_{\mathcal{K}}$ denote the identity operators on \mathcal{H} , \mathcal{K} respectively.

A $\text{Bor}(X)$ -measurable complex function $f : X \rightarrow \mathbb{C}$ is integrable with respect to a spectral measure $M_P : \text{Bor}(X) \rightarrow W_2$ with $P \in (W_1)_p$, if there exists a constant $K_f \in \mathbb{R}$, such that for every $k \in \mathcal{K}$ we have $\int_X |f| d(M_P)_{k, k} \leq K_f \|k\|^2$. A function f is *M -integrable*, if it is M_P -integrable for every $P \in (W_1)_p$. Then the integral of f with respect to a positive operator-valued measure M_A for $A \in (W_1)_+$ is defined by $\int_X f dM_A := \int_X f dM_{S_\ell(A)}$ for a limiting sequence $S_\ell(A)$ of A (it is independent from the choice of the limiting sequence). Finally, the integral $\int_X f dM_A$ of f with respect to a signed operator-valued measure M_A for $A \in W_1$ is defined by $\int_X f dM_{\text{Re}(A)_+} - \int_X f dM_{\text{Re}(A)_-} + i \cdot \int_X f dM_{\text{Im}(A)_+} - i \cdot \int_X f dM_{\text{Im}(A)_-}$, where $\text{Re}(A)$, $\text{Im}(A)$ are the real and the imaginary part of A and A_+ , A_- are the positive and the negative part of A . The set of all M -integrable functions is denoted by $\mathcal{I}(M)$. The map $\mathcal{B} : \mathcal{I}(M) \times W_1 \rightarrow W_2$, defined by $\mathcal{B}(f, A) := \int_X f dM_A$, is bilinear. \mathcal{B} extends to a linear map $\overline{\mathcal{B}} : \mathcal{I}(M) \otimes W_1 \rightarrow W_2$ and we define an *integral* of $F \in \mathcal{I}(M) \otimes W_1$ with respect to M by $\int_X F dM := \overline{\mathcal{B}}(F)$. The algebraic properties of an integral with respect to M are given in the following proposition (see [16, Proposition 3.5] and [16, Proposition 7.2]).

Proposition 2.4. *For $F, G \in \mathcal{I}(M) \otimes W_1$, $A \in W_1$ and $\lambda \in \mathbb{C}$ we have:*

- (1) $\int_X (F + G) dM = \int_X F dM + \int_X G dM$.
- (2) $\int_X \lambda F dM = \lambda \int_X F dM$.
- (3) *For every $\Delta \in \text{Bor}(X)$, $\int_X \chi_\Delta \otimes A dM = M_A(\Delta)$.*
- (4) *From $F \succeq 0$, it follows that $\int_X F dM \succeq 0$.*
- (5) $\int_X FG dM = (\int_X F dM)(\int_X G dM)$.

2.2. Spectral integral on a Hilbert space. We recall the integration of unbounded functions with respect to the spectral measure (see [15]). Let X be a set, $\text{Bor}(X)$ a Borel σ -algebra on X , \mathcal{H} is a Hilbert space and $E : \text{Bor}(X) \rightarrow B(\mathcal{H})$ a spectral measure. Let \mathcal{U} be the set of all $\text{Bor}(X)$ -measurable functions $f : X \rightarrow \mathbb{C}$. A sequence $(\Delta_n)_{n \in \mathbb{N}}$ of sets $\Delta_n \in \text{Bor}(X)$ is a *bounding sequence* for a subset \mathcal{F} of \mathcal{U} if each function $f \in \mathcal{F}$ is bounded on Δ_n , $\Delta_n \subseteq \Delta_{n+1}$ for $n \in \mathbb{N}$, and $E(\bigcup_{n=1}^\infty \Delta_n) = \text{Id}_{\mathcal{H}}$.

If $(\Delta_n)_{n \in \mathbb{N}}$ is any bounding sequence, then by the properties of the spectral measure, $E(\Delta_n) \preceq E(\Delta_{n+1})$ for $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} E(\Delta_n)x = x$ for $x \in \mathcal{H}$. Each finite set of functions $f_1, f_2, \dots, f_r \in \mathcal{U}$ has a bounding sequence $\Delta_n := \{t \in X : |f_j(t)| \leq n \text{ for } j = 1, 2, \dots, r\}$.

The spectral integral $\mathbb{I}(f) := \int_X f \, dE$ of a function $f \in \mathcal{U}$ is given by the following theorem (see [15, Theorem 4.13]).

Theorem 2.5. *Suppose that $f \in \mathcal{U}$ and define*

$$\mathcal{D}(\mathbb{I}(f)) := \{x \in \mathcal{H} : \int_X |f(t)|^2 \, d\langle E(t)x, x \rangle < \infty\}.$$

Let $(\Delta_n)_{n \in \mathbb{N}}$ be a bounding sequence for f . Then we have:

- (1) *A vector $x \in \mathcal{H}$ is in $\mathcal{D}(\mathbb{I}(f))$ iff the sequence $(\mathbb{I}(f\chi_{\Delta_n})x)_{n \in \mathbb{N}}$ converges in \mathcal{H} , or equivalently, if $\sup_{n \in \mathbb{N}} \|\mathbb{I}(f\chi_{\Delta_n})x\| < \infty$.*
- (2) *For $x \in \mathcal{D}(\mathbb{I}(f))$, the limit of the sequence $(\mathbb{I}(f\chi_{\Delta_n})x)$ does not depend on the bounding sequence (Δ_n) . There is a linear operator $\mathbb{I}(f)$ on $\mathcal{D}(\mathbb{I}(f))$ defined by $\mathbb{I}(f)x = \lim_{n \rightarrow \infty} \mathbb{I}(f\chi_{\Delta_n})x$ for $x \in \mathcal{D}(\mathbb{I}(f))$.*
- (3) *The vector space $\cup_{n=1}^{\infty} E(\Delta_n)\mathcal{K}$ is contained in $\mathcal{D}(\mathbb{I}(f))$ and is a core for $\mathbb{I}(f)$. Furthermore, $E(\Delta_n)\mathbb{I}(f) \subseteq \mathbb{I}(f)E(\Delta_n) = \mathbb{I}(f\chi_{\Delta_n})$.*

$\mathcal{D}(\mathbb{I}(f))$ from Theorem 2.5 is the domain of the operator $\mathbb{I}(f)$. By \overline{A} we denote the closure of a closable linear operator A , defined on a dense subspace of a Hilbert space \mathcal{H} . The main algebraic properties of the map $f \rightarrow \mathbb{I}(f)$ are given in the following theorem (see [15, Theorem 4.16]).

Theorem 2.6. *For $f, g \in \mathcal{U}$ and $\alpha, \beta \in \mathbb{C}$ we have:*

- (1) $\mathbb{I}(\overline{f}) = \mathbb{I}(f)^*$,
- (2) $\mathbb{I}(\alpha f + \beta g) = \overline{\alpha \mathbb{I}(f) + \beta \mathbb{I}(g)}$,
- (3) $\mathbb{I}(fg) = \mathbb{I}(f)\mathbb{I}(g)$,
- (4) $\mathbb{I}(f)$ is a normal operator on \mathcal{K} , and $\mathbb{I}(f)^*\mathbb{I}(f) = \mathbb{I}(\overline{f}f) = \mathbb{I}(f)\mathbb{I}(f)^*$,
- (5) $\mathcal{D}(\mathbb{I}(f)\mathbb{I}(g)) = \mathcal{D}(\mathbb{I}(g)) \cap \mathcal{D}(\mathbb{I}(fg))$.

To emphasize with respect to which spectral measure we integrate, we will denote the integral $\mathbb{I}(f)$ with respect to E by $\mathbb{I}_E(f)$.

2.3. *-representations of commutative semigroups with an involution. In this subsection we present integral representation theorems [10, Theorem 2] and [11, Theorem 1.2.] for *-representations of a commutative semigroup S with identity element e and an involution $*$ (i.e., $(s^*)^* = s$ and $(st)^* = s^*t^*$ for all $s, t \in S$). A function $\chi : S \rightarrow \mathbb{C}$ which satisfies $\chi(e) = 1$ and $\chi(st^*) = \chi(s)\overline{\chi(t)}$ for all $s, t \in S$ is called a *character* of S . By S^* we denote the set of all characters of S . The set S^* is a completely regular space when equipped with the topology of pointwise convergence from \mathbb{C}^S . Let \mathcal{H} be a Hilbert space and $W \subseteq B(\mathcal{H})$. The map $\rho : S \rightarrow W$ is a *-representation of S if $\rho(e) = \text{Id}_{\mathcal{H}}$ and $\rho(st^*) = \rho(s)\rho(t)^*$ for all $s, t \in S$. For $s \in S$, the function $f_s : S^* \rightarrow \mathbb{C}$ is defined by $f_s(\chi) = \chi(s)$. [10, Theorem 2] is the following result.

Theorem 2.7. *Let $\rho : S \rightarrow W$ be a *-representation of a semigroup S . Then there exists a unique regular spectral measure $E : \text{Bor}(S^*) \rightarrow W$ with a compact support, such that $\rho(s) = \int_{S^*} f_s(\chi) \, dE(\chi)$ for all $s \in S$.*

Remark 2.8. Originally, in [10, Theorem 2], a von Neumann algebra W is $B(\mathcal{H})$. Now we explain, why we can replace $B(\mathcal{H})$ by W . By [10, Theorem 2], $\rho : S \rightarrow W \subseteq B(\mathcal{H})$ is represented by a unique regular spectral measure $E : \text{Bor}(S^*) \rightarrow B(\mathcal{H})$ with a compact support K . Define the sets $\mathcal{A}_K = \{(f_b)|_K \in C(K) : b \in \mathcal{B}\}$ and $B_{\mathcal{A}_K} := \{f \in \mathcal{A}_K : \|f\|_{\infty} \leq 1\}$. By [3, p. 95], $B_{\mathcal{A}_K}$ is dense in the ball of $C(K)$

equipped with a supremum norm. Therefore, ρ can be extended by the continuity to the $*$ -representation $\tilde{\rho} : C(K) \rightarrow W$. By Theorem 1.1, $\tilde{\rho}$ is represented by a unique regular spectral measure $\tilde{E} : \text{Bor}(K) \rightarrow W$. From the uniqueness, $E = \tilde{E}$ and thus, $E : \text{Bor}(S^*) \rightarrow W$ maps into W as desired.

Now we explain the extension of Theorem 2.7 to a $*$ -representation with a range in (not necessarily bounded) normal operators. A function $\alpha : S \rightarrow [0, \infty)$ is an *absolute value* if $\alpha(s^*) = \alpha(s)$, $\alpha(e) = 1$, and $\alpha(st) \leq \alpha(s)\alpha(t)$ for all $s, t \in S$. The family of all absolute values is denoted by $\mathcal{A}(S)$. Given a map $\rho : S \rightarrow \mathcal{N}(\mathcal{H})$, where $\mathcal{N}(\mathcal{H})$ is a vector space of all normal (not necessarily bounded) operators on a Hilbert space \mathcal{H} and $\alpha \in \mathcal{A}(S)$, define the set $D_\alpha := \{h \in \bigcap_{s \in S} \mathcal{D}(\rho(s)) : \|\rho(s)h\| \leq \alpha(s)\|h\| \text{ for all } s \in S\}$, where $\mathcal{D}(\rho(s))$ denotes the domain of $\rho(s)$. By [11, Definition 1.1.], a map $\rho : S \rightarrow \mathcal{N}(\mathcal{H})$ is called a *$*$ -representation*, if:

- (1) $\rho(e) = \text{Id}_{\mathcal{H}}$, where $\text{Id}_{\mathcal{H}}$ is the identity operator on \mathcal{H} .
- (2) $\rho(s^*) = \rho(s)^*$, $s \in S$.
- (3) $\overline{\rho(t)\rho(s)} \subseteq \rho(st)$ with $\mathcal{D}(\rho(t)\rho(s)) = \mathcal{D}(\rho(st)) \cap \mathcal{D}(\rho(s))$, $s, t \in S$.
- (4) $\overline{\rho(t)\rho(s)} = \rho(st)$, $s, t \in S$.
- (5) $D_c := \bigcup_{\alpha \in \mathcal{A}(S)} D_\alpha$ is dense in \mathcal{H} .

[11, Theorem 1.2.] is the following result.

Theorem 2.9. *Assume the notation above. Let $\rho : S \rightarrow \mathcal{N}(\mathcal{H})$ be a $*$ -representation. Then there exists a unique regular spectral measure $E : \text{Bor}(S^*) \rightarrow B(\mathcal{H})$, such that $\text{supp}(E_{h,h})$ is compact iff $h \in D_c$ and $\rho(s)x = \int_{S^*} f_s(\chi) dE(\chi) x$ for every $x \in \mathcal{D}(\rho(s))$ and all $s \in S$.*

3. PROOFS OF THEOREMS A AND B

Proof of Theorem A. The non-trivial direction is (1) \Rightarrow (2). In particular, \mathcal{B} is a commutative semigroup with an involution. By Theorem 2.7, there exists a unique regular spectral measure $\tilde{E} : \text{Bor}(\mathcal{B}^*) \rightarrow W$ with a compact support such that $\rho(b) = \int_{\mathcal{B}^*} f_b(\chi) d\tilde{E}(\chi)$ for every $b \in \mathcal{B}$. By the proof of [10, Theorem 1], since \mathcal{B} is induced from an algebra, the support of \tilde{E} is contained in the set of linear characters of \mathcal{B} . Hence, $E : \text{Bor}(\mathcal{B}) \rightarrow W$, defined by $E(\Delta) = \tilde{E}(\Delta)$, satisfies the statement of Theorem A. \square

Assume the notation as in Theorem B. For every $b \in \mathcal{B}$, we define the linear operator $\rho_b : W_1 \rightarrow W_2$ by $\rho_b(A) := \rho(b \otimes A)$. We notice the following.

Proposition 3.1. *ρ_b is a bounded linear operator for every $b \in \mathcal{B}$.*

Proof. Let us write $b = \frac{(b+1)^*(b+1)}{2} - \frac{(b-1)^*(b-1)}{2}$. Denote $c = \frac{(b+1)(b+1)^*}{2}$ and $d = \frac{(b-1)(b-1)^*}{2}$. Then $\rho_b = \rho_c - \rho_d$. Thus it suffices to prove, that ρ_b is bounded for every $b \in \mathcal{B}^2 := \{a^*a : a \in \mathcal{B}\}$. Since ρ is a $*$ -representation, $\rho_b : W_1 \rightarrow W_2$ is a positive linear operator for every $b = a^*a \in \mathcal{B}^2$. Indeed,

$$\begin{aligned} \rho_b(A) &= \rho(b \otimes A) = \rho(a^*a \otimes A^{\frac{1}{2}}A^{\frac{1}{2}}) = \rho((a \otimes A^{\frac{1}{2}})^*(a \otimes A^{\frac{1}{2}})) \\ &= \rho((a \otimes A^{\frac{1}{2}})^*)\rho(a \otimes A^{\frac{1}{2}}) = \rho(a \otimes A^{\frac{1}{2}})^*\rho(a \otimes A^{\frac{1}{2}}) \in (W_2)_+ \end{aligned}$$

where $A^{\frac{1}{2}}$ is a positive square root of A . For every k in the Hilbert space \mathcal{K} , where $W_2 \subseteq B(\mathcal{K})$, $(\rho_b)_k : W_1 \rightarrow \mathbb{C}$, defined by $(\rho_b)_k(A) := \langle \rho(b \otimes A)k, k \rangle$, is a positive linear functional. By [5, 5.12. Corollary], it is bounded and has norm $(\rho_b)_k(\text{Id}_{\mathcal{H}}) =$

$\langle \rho(b \otimes \text{Id}_{\mathcal{H}})k, k \rangle$. Hence, $|(\rho_b)_k(A)| \leq (\rho_b)_k(\text{Id}_{\mathcal{H}})\|A\| \leq \|\rho(b \otimes \text{Id}_{\mathcal{H}})\| \|k\|^2 \|A\|$ for every $A \in W_1$. Therefore,

$$\|\rho_b(A)\| = \sup_{\|k\|=1} |(\rho_b)_k(A)| \leq \sup_{\|k\|=1} \|\rho(b \otimes \text{Id}_{\mathcal{H}})\| \|k\|^2 \|A\| = \|\rho(b \otimes \text{Id}_{\mathcal{H}})\| \|A\|$$

for every $A \in W_1$. Hence, ρ_b is a bounded linear operator with norm $\|\rho(b \otimes \text{Id}_{\mathcal{H}})\|$. \square

Essential technical lemma in the proof of Theorem B is the following.

Lemma 3.2. *Let $K \in \text{Bor}(\widehat{\mathcal{B}})$ be a compact set. Let us equip the algebra $\mathcal{A}_K = \{(f_b)|_K \in C(K) : b \in \mathcal{B}\}$ of continuous functions with a supremum norm. Suppose $M(K)$ is the normed space of all complex-valued regular Borel measures on K equipped with a variation norm. Then the unit ball $B_{\mathcal{A}_K} := \{f \in \mathcal{A}_K : \|f\|_{\infty} \leq 1\}$ is dense in the unit ball of $C(K)^{**} = M(K)^*$ equipped with a weak*-topology.*

Proof. By [5, V.4.1. Proposition], the unit ball of $C(K)$ is dense in $M(K)^*$ equipped with a weak*-topology. By [3, p. 95], $B_{\mathcal{A}_K}$ is dense in the ball of $C(K)$. \square

Proof of Theorem B. The non-trivial direction is (1) \Rightarrow (2). Since ρ is a *-representation, the maps $\rho_P : \mathcal{B} \rightarrow W_2$, $\rho_P(b) := \rho(b \otimes P)$ are *-representations for every $P \in (W_1)_p$. By Theorem A, there exist unique spectral measures $E_P : \text{Bor}(\widehat{\mathcal{B}}) \rightarrow W_2$ with a compact support, such that $\rho_P(b) = \int_{\widehat{\mathcal{B}}} f_b(\chi) dE_P(\chi)$ holds for every $b \in \mathcal{B}$ and every $P \in (W_1)_p$. The idea is to show that the family $\{E_P\}_{P \in (W_1)_p}$ satisfies the conditions of Theorem 2.2 to obtain a non-negative spectral measure M representing ρ .

First let us show, that $\text{supp}(E_P) \subseteq \text{supp}(E_{\text{Id}_{\mathcal{H}}})$ for every $P \in (W_1)_p$. Define $K := \text{supp}(E_P) \cup \text{supp}(E_{\text{Id}_{\mathcal{H}}})$. Let us assume that $\text{supp}(E_P) \setminus \text{supp}(E_{\text{Id}_{\mathcal{H}}}) \neq \emptyset$ for some $P \in (W_1)_p$. Hence, there are a set $\Delta \in \text{supp}(E_P) \setminus \text{supp}(E_{\text{Id}_{\mathcal{H}}})$ and a vector $k_1 \in \mathcal{K}$ such that $(E_P)_{k_1, k_1}(\Delta) > 0$ and $(E_{\text{Id}_{\mathcal{H}}})_{k_1, k_1}(\Delta) = 0$. By Lemma 3.2, there exists a net $f_{b_j} \in C(K)$, such that $\int_K f_{b_j} d(E_P)_{k_1, k_1} \rightarrow (E_P)_{k_1, k_1}(\Delta) > 0$ and $\int_K f_{b_j} d(E_{\text{Id}_{\mathcal{H}}-P})_{k_1, k_1} \rightarrow (E_{\text{Id}_{\mathcal{H}}-P})_{k_1, k_1}(\Delta) \geq 0$. However, for every j in the net we have

$$\begin{aligned} 0 &= \int_K f_{b_j} d(E_{\text{Id}_{\mathcal{H}}})_{k_1, k_1} = \langle \rho(b_j \otimes \text{Id}_{\mathcal{H}})k_1, k_1 \rangle = \\ &= \langle \rho(b_j \otimes P)k_1, k_1 \rangle + \langle \rho(b_j \otimes (\text{Id}_{\mathcal{H}} - P))k_1, k_1 \rangle = \\ &= \int_K f_{b_j} d(E_P)_{k_1, k_1} + \int_K f_{b_j} d(E_{\text{Id}_{\mathcal{H}}-P})_{k_1, k_1}. \end{aligned}$$

Since $\lim_j \int_K f_{b_j} d(E_P)_{k_1, k_1} > 0$ and $\lim_j \int_K f_{b_j} d(E_{\text{Id}_{\mathcal{H}}-P})_{k_1, k_1} \geq 0$, this is a contradiction.

It remains to check first that the family $\{E_P\}_{P \in (W_1)_p}$ satisfies the conditions of Theorem 2.2, second that M is a representing measure of ρ and finally that M is unique, regular and normalized. Since the support of every E_P is contained in the compact set $K = \text{supp}(E_{\text{Id}_{\mathcal{H}}})$, the proofs are the same as the proof of direction (1) \Rightarrow (2) of [16, Theorem 9.1], just that we replace the use of [16, Lemma 2.3] by Lemma 3.2 above. \square

4. PROOF OF THEOREM C'

A *bounding sequence* E_n of a closed operator A in a Hilbert space \mathcal{K} is an increasing family of hermitian projections, such that $\cup_{n=1}^{\infty} E_n = \text{Id}_{\mathcal{K}}$ and $E_n A \subseteq A E_n$ and $A E_n$ is bounded everywhere defined operator on \mathcal{K} for every $n \in \mathbb{N}$.

Proof of Theorem C'. The direction (2) \Rightarrow (1) follows by Propositions 1.3 and 1.4. Now we will prove the direction (1) \Rightarrow (2). In particular, \mathcal{B} is a commutative semigroup with an involution, where the group operation is the algebra multiplication. Let $\mathcal{N}(\mathcal{H})$ be the set of normal operators on a Hilbert space \mathcal{H} . First we prove, that the representation $\rho_1 : \mathcal{B} \rightarrow \mathcal{N}(\mathcal{H})$, $\rho_1(b) := \overline{\rho(b)}$, is well-defined and satisfies the conditions (1)-(5) in the definition of a $*$ -representation of a semigroup with an involution as defined in Subsection 2.3.

Well-definedness: $\rho_1(b)$ has to be a normal operator for every $b \in \mathcal{B}$. Since ρ is integrable, this is true by [14, Theorem 9.1.2].

Condition (1): By the definition of a $*$ -representation, $\rho(1) = \text{Id}_{\mathcal{D}}$ and hence $\rho_1(1) = \text{Id}_{\mathcal{H}}$.

Condition (2): By the definition of integrability, $\overline{\rho(b^*)} = \rho(b)^*$. Since $\rho(b)$ is closable, $\rho(b)^* = \overline{\rho(b)^*}$. Thus, $\rho_1(b^*) = \rho_1(b)^*$.

Condition (3): The following chain is true.

$$\begin{aligned} \overline{\rho(a)} \overline{\rho(b)} &= \overline{\rho(a^{**})} \overline{\rho(b^{**})} = \rho(a^*)^* \rho(b^*)^* \subseteq (\rho(b^*) \rho(a^*))^* \\ &= \rho(b^* a^*)^* = \overline{\rho((b^* a^*)^*)} = \overline{\rho(ab)}, \end{aligned}$$

where the second equalities in both lines follow by integrability of ρ . Hence, $\rho_1(a) \rho_1(b) \subseteq \rho_1(ab)$. and $\mathcal{D}(\rho_1(a) \rho_1(b)) \subseteq \mathcal{D}(\rho_1(b)) \cap \mathcal{D}(\rho_1(ab))$. To prove the converse inclusion, let $h \in \mathcal{D}(\rho_1(b)) \cap \mathcal{D}(\rho_1(ab))$. By the integrability of ρ , there is an abelian von Neumann algebra \mathcal{N} such that $\rho_1(a)$ is affiliated with \mathcal{N} for all $a \in \mathcal{B}$ (see [14, Theorem 9.1.7]). By [6, Theorem 5.6.15], $\rho_1(a)$, $\rho_1(b)$ and $\rho_1(ab)$ have a common bounding sequence E_n . Since $h \in \mathcal{D}(\rho_1(b))$, we have $E_n \rho_1(b) h = \rho_1(b) E_n h$. Hence, $\rho_1(a) \rho_1(b) E_n h = \rho_1(a) E_n (\rho_1(b) h)$ is well-defined. By, $\rho_1(a) \rho_1(b) \subseteq \rho_1(ab)$, we have $\rho_1(ab) E_n h = \rho_1(a) \rho_1(b) E_n h$ for every $n \in \mathbb{N}$. As $n \rightarrow \infty$, it follows that $\rho_1(b) E_n h \rightarrow \rho_1(b) h$ (by $h \in \mathcal{D}(\rho_1(b))$) and $\rho_1(ab) E_n h \rightarrow \rho_1(ab) h$ (by $h \in \mathcal{D}(\rho_1(ab))$). Therefore, since the operator $\rho_1(a)$ is closed, it follows that $\rho_1(b) h \in \mathcal{D}(\rho_1(a))$. So, $h \in \mathcal{D}(\rho_1(a) \rho_1(b))$. Hence, $\mathcal{D}(\rho_1(a) \rho_1(b)) = \mathcal{D}(\rho_1(ab)) \cap \mathcal{D}(\rho_1(b))$. Therefore, (iii) is true.

Condition (4): By (iii), we have $\overline{\rho_1(a) \rho_2(b)} \subseteq \rho_1(ab)$. We also know, that

$$\rho_1(ab) = \overline{\rho(ab)} = \overline{\rho(a) \rho(b)} \subseteq \overline{\overline{\rho(a)} \overline{\rho(b)}} = \overline{\rho_1(a) \rho_1(b)}.$$

Thus, $\rho_1(ab) \subseteq \overline{\rho_1(a) \rho_2(b)}$. Hence, $\overline{\rho_1(a) \rho_1(b)} = \rho_1(ab)$. This is exactly the condition (4).

Condition (5): It is fulfilled by the assumption.

Thus, by Theorem 2.9 above, there exists a unique regular spectral measure $E : \text{Bor}(\mathcal{B}^*) \rightarrow B(\mathcal{K})$ such that $\overline{\rho(b)} x = \int_{\mathcal{B}^*} f_b(\chi) dE(\chi) x$, for every $b \in \mathcal{B}$ and every $x \in \mathcal{D}(\overline{\rho(b)})$. Since \mathcal{B} is also an algebra, E -almost every character $\chi \in \mathcal{B}^*$ is linear (see [10, Proof of Theorem 1, p. 2953] and [11, Proof of Theorem 1.2, p. 230]). Therefore, $\overline{\rho(b)} x = \int_{\mathcal{B}} f_b(\chi) dE(\chi) x$ for every $x \in \mathcal{D}(\overline{\rho(b)})$. \square

5. INTEGRAL OF UNBOUNDED FUNCTIONS WITH RESPECT TO A NON-NEGATIVE SPECTRAL MEASURE

Let $(X, \text{Bor}(X), W \subseteq B(\mathcal{H}), B(\mathcal{K}), M)$ be a space with a non-negative spectral measure M , where X is a topological space, $\text{Bor}(X)$ a Borel σ -algebra on X , $W \subseteq B(\mathcal{H})$ a von Neumann algebra and $B(\mathcal{H}), B(\mathcal{K})$ the bounded linear operators on Hilbert spaces \mathcal{H}, \mathcal{K} .

Before solving Problem D, the definition of the integration of functions with respect to a non-negative spectral measure $M : \text{Bor}(X) \rightarrow B(W, B(\mathcal{K}))$ has to be extended to a larger set of functions. In [16], a measurable function f was M -integrable, if it was M_A -essentially bounded for every positive operator $A \in W_+$, i.e., there exists a constant K_f such that $\langle (\int_X |f| dM_A)k, k \rangle \leq K_f \|k\|^2$ for every $k \in \mathcal{K}$. In this section we extend this definition to functions, which do not necessarily satisfy this condition.

Let $\mathcal{K} := \{K \subseteq X : K \text{ compact}\}$ be the set of all compact subsets of X . Let us define the set $\mathcal{D}_0 \subseteq \mathcal{K}$ by

$$\mathcal{D}_0 := \bigcup_{K \in \mathcal{K}} M(K)(\text{Id}_{\mathcal{H}})\mathcal{K}.$$

Proposition 5.1. \mathcal{D}_0 is a dense linear subspace in \mathcal{K} in either of the following cases:

- The topological space X is σ -compact.
- The measure M is regular.

Proof. First we show \mathcal{D}_0 is a linear subspace in \mathcal{K} . Let $x_1, x_2 \in \mathcal{D}_0$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. Then it holds that $x_1 = M(K_1)(\text{Id}_{\mathcal{H}})k_1$, $x_2 = M(K_2)(\text{Id}_{\mathcal{H}})k_2$ for some compact sets K_1, K_2 and some vectors $k_1, k_2 \in \mathcal{K}$. Since $M(\Delta)(\text{Id}_{\mathcal{H}})$ is a hermitian projection for every $\Delta \in \text{Bor}(X)$ it follows that $M(K_i)(\text{Id}_{\mathcal{H}})x_i = (M(K_i)(\text{Id}_{\mathcal{H}}))^2 k_i = M(K_i)(\text{Id}_{\mathcal{H}})x_i = x_i$ for $i = 1, 2$. By the inequality $M(\Delta_1)(\text{Id}_{\mathcal{H}}) \preceq M(\Delta_2)(\text{Id}_{\mathcal{H}})$ if $\Delta_1 \subseteq \Delta_2$, it follows that $\lambda_1 x_1 + \lambda_2 x_2 = M(K_1 \cup K_2)(\text{Id}_{\mathcal{H}})(\lambda_1 x_1 + \lambda_2 x_2)$. Hence, $\lambda_1 x_1 + \lambda_2 x_2 \in \mathcal{D}_0$ and \mathcal{D}_0 is a linear subspace in \mathcal{K} .

Now we prove \mathcal{D}_0 is dense in \mathcal{K} . We separate two cases:

- Let us assume X is σ -compact. Take $k \in \mathcal{K}$. By $M(X)(\text{Id}_{\mathcal{H}})k = k$ and by the σ -compactness of X , we have $k = M(X)(\text{Id}_{\mathcal{H}})k = \lim_{n \in \mathbb{N}} M(K_n)(\text{Id}_{\mathcal{H}})k$, where K_n is an increasing sequence of compact sets, such the $\cup_n K_n = X$. Hence, \mathcal{D}_0 is dense.
- Let now X be arbitrary and M regular. Since $(M_{\text{Id}_{\mathcal{H}}})_{k,k}$ is regular for every $k \in \mathcal{H}$, $\sup_{K \in \mathcal{K}} (M_{\text{Id}_{\mathcal{H}}})_{k,k}(K) = (M_{\text{Id}_{\mathcal{H}}})_{k,k}(X) = \|k\|^2$. Then

$$\begin{aligned} \|k - M_{\text{Id}_{\mathcal{H}}}(K)k\| &= \|M_{\text{Id}_{\mathcal{H}}}(K^c)k\| = ((M_{\text{Id}_{\mathcal{H}}})_{k,k}(K^c))^{\frac{1}{2}} \\ &= ((M_{\text{Id}_{\mathcal{H}}})_{k,k}(X) - (M_{\text{Id}_{\mathcal{H}}})_{k,k}(K))^{\frac{1}{2}} = (\|k\|^2 - (M_{\text{Id}_{\mathcal{H}}})_{k,k}(K))^{\frac{1}{2}}, \end{aligned}$$

is true for every $K \in \mathcal{K}$ and hence, $\sup_{K \in \mathcal{K}} M_{\text{Id}_{\mathcal{H}}}(K)k = k$ □

From now on we will assume that the space \mathcal{D}_0 is dense. We say a $\text{Bor}(X)$ -measurable function f is M -integrable if \mathcal{D}_0 is in the domain of the operator $\int_X f dM_P$ for every hermitian projection $P \in (W)_p$. By $\mathcal{I}(M)$ we denote be the set of all M -integrable function.

Proposition 5.2. *The set $\mathcal{I}(M)$ is a complex vector space and it contains all bounded functions on X and all continuous functions on X .*

Proof. The fact that $\mathcal{I}(M)$ is a complex vector space is clear. Let now f be a bounded function on X or a continuous functions on X . Pick $k \in \mathcal{D}_0$. Then there are a compact set $K \in \mathcal{K}$ and $k' \in \mathcal{K}$ such that $M(K)(\text{Id}_{\mathcal{H}})k' = k$. Hence, $M(K)(\text{Id}_{\mathcal{H}})k = M(K)(\text{Id}_{\mathcal{H}})^2 k' = M(K)(\text{Id}_{\mathcal{H}})k' = k$. Thus, by the compactness of K and $(M_{\text{Id}_{\mathcal{H}}})_{k,k}$ being a finite measure, it holds that $\int_X |f|^2 d(M_{\text{Id}_{\mathcal{H}}})_{k,k} = \int_K |f|^2 d(M_{\text{Id}_{\mathcal{H}}})_{k,k} < \infty$. Since $(M_{\text{Id}_{\mathcal{H}}})_{k,k} = (M_P)_{k,k} + (M_{\text{Id}_{\mathcal{H}}-P})_{k,k}$, k also belongs to $\mathbb{I}_{M_P}(f)$ for every hermitian projection $P \in W_p$. Hence, $\mathcal{D}_0 \subseteq \mathbb{I}_{M_P}(f)$. \square

Construction of an integral \mathbb{I}_M . Now we explain the construction of the integral with respect to a non-negative spectral measure, denoted by \mathbb{I}_M , and prove well-definedness in what follows.

Step 1 - The construction of a preintegral $\psi : \mathcal{I}(M) \times W \rightarrow \mathcal{L}^+(\mathcal{D}_0)$:

- For $(f, P) \in \mathcal{I}(M) \times W_p$, we define $\psi(f, P) := (\int_X f dM_P)|_{\mathcal{D}_0}$.
- For $(f, A) \in \mathcal{I}(M) \times W_+$ such that A has a finite spectral decomposition $\sum_{k=1}^n \lambda_k P_k$, where $\lambda_k \geq 0$ are non-negative and P_k are mutually orthogonal hermitian projections (i.e., $P_i P_j = 0$ for every $i \neq j$), we define $\psi(f, A) := \sum_{k=1}^n \lambda_k \psi(f, P_k)$.
- For $(f, A) \in \mathcal{I}(M) \times W_+$ such that A does not have a finite spectral decomposition, take an arbitrary limiting sequence $S_\ell(A)$ and define $\psi(f, A) := \lim_{\ell \rightarrow \infty} \psi(f, S_\ell(A))$. By Proposition 5.4 below, $\psi(f, A)$ is well-defined and does not depend on the choice of the sequence $S_\ell(A)$.
- For $(f, A) \in \mathcal{I}(M) \times W$, define

$$\psi(f, \text{Re}(A)_+)x - \psi(f, \text{Re}(A)_-)x + i \cdot \psi(f, \text{Im}(A)_+)x - i \cdot \psi(f, \text{Im}(A)_-)x,$$

where $\text{Re}(A), \text{Im}(A)$ denote the real and the imaginary part of the operator A and A_+, A_- the positive and the negative part of the hermitian operator A .

- ψ is a bilinear form (see Proposition 5.6).

Step 2 - Defining a map $\tilde{\psi} : \mathcal{I}(M) \otimes W \rightarrow \mathcal{L}^+(\mathcal{D}_0)$:

Since ψ is a bilinear form, it extends, by the universal property of tensor products, to the linear map $\tilde{\psi} : \mathcal{I}(M) \otimes W \rightarrow \mathcal{L}^+(\mathcal{D}_0)$, defined by $\tilde{\psi}(\sum_{i=1}^n f_i \otimes A_i) = \sum_{i=1}^n \psi(f_i, A_i)$.

Step 3 - Defining an integral \mathbb{I}_M :

We extend $\tilde{\psi}$ to the integral $\mathbb{I}_M : \mathcal{I}(M) \otimes W \rightarrow \mathcal{N}(\mathcal{K})$, defined by the rule $\mathbb{I}_M(\sum_{i=1}^n f_i \otimes A_i) = \overline{\tilde{\psi}(\sum_{i=1}^n f_i \otimes A_i)}$, where $\mathcal{N}(\mathcal{K})$ denotes the vector space of normal operators on \mathcal{K} and $\overline{}$ denotes the closure of a densely defined operator $T \in \mathcal{L}^+(\mathcal{D}_0)$. By Proposition 5.8, \mathbb{I}_M is well-defined.

It remains to prove Propositions 5.4, 5.6 and 5.8 which were needed for the construction of \mathbb{I}_M . First we prove the following lemma, which will be needed in the proofs.

Lemma 5.3. *Let $f \in \mathcal{I}(M)$. Then \mathcal{D}_0 is a core for $\mathbb{I}_{M_P}(f)$. Also,*

$$M(K)(\text{Id}_{\mathcal{H}})\mathbb{I}_{M_P}(f) \subseteq \mathbb{I}_{M_P}(f)M(K)(\text{Id}_{\mathcal{H}}) = \mathbb{I}_{M_P}(f)\chi_K$$

for every compact set K .

Proof. Let $x \in \mathcal{K}$ be arbitrary. Let Δ_n be the bounding sequence of f . By the boundedness of the function $f\chi_{\Delta_n}\chi_{\Delta_n}$ and by [16, Proposition 7.2], we have the following chain of equalities $(\int_X f\chi_{\Delta_n} \otimes P \, dM)x = (\int_X f\chi_{\Delta_n}\chi_{\Delta_n} \otimes P \, dM)x = (\int_X f\chi_{\Delta_n} \otimes P \, dM)M(\Delta_n)(\text{Id}_{\mathcal{H}})x = M(\Delta_n)(\text{Id}_{\mathcal{H}})(\int_X f\chi_{\Delta_n} \otimes P \, dM)x$, where m is greater than n . Letting $m \rightarrow \infty$, we conclude that $(\int_X f\chi_{\Delta_n} \otimes P \, dM)x = (\int_X f \otimes P \, dM)M(\Delta_n)(\text{Id}_{\mathcal{H}})x$. For $x \in \mathcal{D}(\mathbb{I}_{M_P}(f))$, again sending $m \rightarrow \infty$, it follows that $(\int_X f \otimes P \, dM)M(\Delta_n)(\text{Id}_{\mathcal{H}})x = M(\Delta_n)(\text{Id}_{\mathcal{H}})(\int_X f \otimes P \, dM)x$. Since $M(\Delta_n)(\text{Id}_{\mathcal{H}})x \rightarrow x$ and $\mathbb{I}_{M_P}(f)M(\Delta_n)(\text{Id}_{\mathcal{H}})x = M(\Delta_n)(\text{Id}_{\mathcal{H}})\mathbb{I}_{M_P}(f)x \rightarrow \mathbb{I}_{M_P}(f)x$ for $x \in \mathcal{D}(\mathbb{I}_{M_P}(f))$, the linear subspace $\cup_{n=1}^{\infty} M(\Delta_n)(\text{Id}_{\mathcal{H}})\mathcal{K}$ is a core for $\mathbb{I}_{M_P}(f)$. By the σ -compactness of X or regularity of M , we have a sequence of compact sets K_n , such that $\|M(\Delta_n)(\text{Id}_{\mathcal{H}})x - M(K_n)x\| \leq \frac{1}{n\|f\|_{\Delta_n, \infty}}$ and $\|(\int_X f\chi_{\Delta_n} \otimes P \, dM)x - (\int_X f\chi_{K_n} \otimes P \, dM)x\| = \|(\int_X f\chi_{\Delta_n \setminus K_n} \otimes P \, dM)x\| \leq \|f\|_{\Delta_n, \infty} \frac{1}{n\|f\|_{\Delta_n, \infty}} = \frac{1}{n}$. Hence, $M(K_n)x \rightarrow x$ and $(\int_X f\chi_{K_n} \otimes P \, dM)x \rightarrow \mathbb{I}_{M_P}(f)x$. Thus, \mathcal{D}_0 is a core for $\mathbb{I}_{M_P}(f)$. \square

Proposition 5.4. *For $f \in \mathcal{I}(M)$ and a positive operator $A \in W_+$ without a finite spectral decomposition, the definition of $\psi(f, A)$:*

- (1) *is well-defined,*
- (2) *does not depend on the choice of the sequence $S_\ell(A)$.*

In the proof of Proposition 5.4 we need the following lemma.

Lemma 5.5. *For $f \in \mathcal{I}(M)$ and $P, Q \in W_p$ orthogonal hermitian projections it is true that:*

- (1) $\psi(f, P + Q) = \psi(f, P) + \psi(f, Q)$.
- (2) $\text{Im}(\psi(f, P)) \perp \text{Im}(\psi(f, Q))$,

where $\text{Im}(T)$ denotes the image of the operator T .

Proof. Since P, Q are orthogonal hermitian projections, $P + Q$ is also a hermitian projection. Since M is a non-negative spectral measure, $\text{Im}(M_P)$ and $\text{Im}(M_Q)$ are orthogonal (Here $\text{Im}(M_P)$, $\text{Im}(M_Q)$ denote the images of M_P , M_Q , i.e., $\text{Im}(M_P) := \bigcup_{\Delta \in \text{Bor}(X)} M_P(\Delta)\mathcal{K}$ and analogously for M_Q). Therefore, by the definition of $\mathcal{D}(\mathbb{I}(f))$ (see Theorem 2.5), $\mathcal{D}(\mathbb{I}_{M_P+Q}(f)) = \mathcal{D}(\mathbb{I}_{M_P}(f)) \cap \mathcal{D}(\mathbb{I}_{M_Q}(f))$. Let Δ_n be a bounding sequence of f (w.r.t. $M_{\text{Id}_{\mathcal{H}}}$ and hence all M_P). Since $f\chi_{\Delta_n}$ is a bounded measurable function, $\mathbb{I}_{M_P+Q}(f\chi_{\Delta_n}) = \mathbb{I}_{M_P}(f\chi_{\Delta_n}) + \mathbb{I}_{M_Q}(f\chi_{\Delta_n})$, by [16, Proposition 3.5]. Hence, by

$$\begin{aligned} \mathbb{I}_{M_P+Q}(f)x &= \lim_{n \rightarrow \infty} \mathbb{I}_{M_P+Q}(f\chi_{\Delta_n})x = \lim_{n \rightarrow \infty} (\mathbb{I}_{M_P}(f\chi_{\Delta_n})x + \mathbb{I}_{M_Q}(f\chi_{\Delta_n})x) \\ &= \lim_{n \rightarrow \infty} \mathbb{I}_{M_P}(f\chi_{\Delta_n})x + \lim_{n \rightarrow \infty} \mathbb{I}_{M_Q}(f\chi_{\Delta_n})x = \mathbb{I}_{M_P}(f)x + \mathbb{I}_{M_Q}(f)x \end{aligned}$$

for every $x \in \mathcal{D}(\mathbb{I}_{M_P+Q}(f))$, it follows that $\mathbb{I}_{M_P+Q}(f) = \mathbb{I}_{M_P}(f) + \mathbb{I}_{M_Q}(f)$ and by the definition of ψ also $\psi(f, P + Q) = \psi(f, P) + \psi(f, Q)$. Since M is a non-negative spectral measure, $M_P(\Delta)M_Q(\Delta') = 0$ for every $\Delta, \Delta' \in \text{Bor}(X)$ such that $\Delta \cap \Delta' = \emptyset$, and hence also $\text{Im}(\psi(f, P)) \perp \text{Im}(\psi(f, Q))$. \square

Proof of Proposition 5.4. Let us first prove that $(\psi(f, S_\ell(A))x)_{\ell \in \mathbb{N}}$ is a Cauchy sequence. For $\ell', \ell \in \mathbb{N}$, $\ell' > \ell$, we have $S_\ell(A) - S_{\ell'}(A) = \sum_{i=1}^{m_{\ell, \ell'}} \lambda_i P_i$ for some $\lambda_i \in \mathbb{R}$, mutually orthogonal hermitian projections P_i and $m_{\ell, \ell'} \in \mathbb{N}$. Given $\epsilon > 0$ and choosing ℓ great enough we can achieve $|\lambda_i| < \epsilon$ for every $i = 1, \dots, m$. Since

$\text{Id}_{\mathcal{H}} = P + (\text{Id}_{\mathcal{H}} - P)$, where $P, \text{Id}_{\mathcal{H}} - P$ are mutually orthogonal hermitian projections, it follows that $\|\psi(f, P)x\| \leq \|\psi(f, \text{Id}_{\mathcal{H}})x\|$ for every $x \in \mathcal{D}_0$. We have

$$\begin{aligned} \left\| \sum_{i=1}^m \lambda_i \psi(f, P_i)x \right\| &\leq \max_i |\lambda_i| \left\| \sum_{i=1}^m \psi(f, P_i)x \right\| \leq \max_i |\lambda_i| \|\psi(f, \text{Id}_{\mathcal{H}})x\| \\ &\leq \epsilon \|\psi(f, \text{Id}_{\mathcal{H}})x\|, \end{aligned}$$

where the first inequality follows by the fact that $\text{Im}(\psi(f, P_i)) \perp \text{Im}(\psi(f, P_j))$ for $i \neq j$ (see Lemma 5.5.(2)) and the second by the fact that $\sum_{i=1}^m P_i$ is a hermitian projection. Since $\epsilon > 0$ was arbitrary, $(\psi(f, S_\ell(A))x)_{\ell \in \mathbb{N}}$ is a Cauchy sequence and hence $\lim_{\ell \rightarrow \infty} \psi(f, S_\ell(A))x$ exists. This proves (1).

Now we will prove the independence from the sequence $S_\ell(A)$. Let $S'_\ell(A) := \sum_{k=1}^{m_\ell} \zeta'_{k,\ell} P'_{k,\ell}$ be another sequence converging to A in norm, where $\zeta'_{k,\ell} \geq 0$ are non-negative and $P'_{k,\ell}$ are mutually orthogonal hermitian projections. We will prove that $(\psi(f, S_\ell(A))x - \psi(f, S'_\ell(A))x)_{\ell \in \mathbb{N}}$ converges to 0. We have $S_\ell(A) - S'_\ell(A) = \sum_{i=1}^p \mu_i Q_i$, where $\mu_i \in \mathbb{R}$ and Q_i are mutually orthogonal hermitian projections. Therefore $\psi(f, S_\ell(A))x - \psi(f, S'_\ell(A))x = \sum_{i=1}^p \mu_i \psi(f, Q_i)x$. Given $\epsilon > 0$ and choosing ℓ great enough we can achieve $|\mu_i| < \epsilon$ for every $i = 1, \dots, p$. As for part (i) we estimate $\|\sum_{i=1}^p \mu_i \psi(f, Q_i)x\| \leq \epsilon \|\psi(f, \text{Id}_{\mathcal{H}})x\|$. Therefore $\psi(f, S_\ell(A))x - \psi(f, S'_\ell(A))x$ converges to 0 which proves (2). \square

Proposition 5.6. *The map ψ is bilinear.*

To prove Proposition 5.6 we need the following lemma.

Lemma 5.7. *For a function $f \in \mathcal{I}(M)$, an operator $A \in W$ and $x \in \mathcal{D}_0$, we have*

$$\psi(f, A)x = \lim_{n \rightarrow \infty} \psi(f \chi_{\Delta_n}, A)x,$$

where $(\Delta_n)_{n \in \mathbb{N}}$ is a bounding sequence of f with respect to the spectral measure $M_{\text{Id}_{\mathcal{H}}}$. Furthermore, for every $n \in \mathbb{N}$ and every $x \in \mathcal{D}_0$ we have

$$\psi(f \chi_{\Delta_n}, A)x = \psi(f, A)M(\Delta_n)(\text{Id}_{\mathcal{H}}).$$

Proof. By the definition of $\psi(f, A)$ for an arbitrary operator $A \in W$, we may assume A is a positive operator. For $f \in \mathcal{I}(M)$, a positive operator $A \in W_+$ and $x \in \mathcal{D}_0$, it holds by the definition that $\psi(f, A)x = \lim_{\ell \rightarrow \infty} \psi(f, S_\ell(A))x$, where $S_\ell(A)$ is a limiting sequence of A . By Theorem 2.5, the equality $\psi(f, P)x = \lim_{n \rightarrow \infty} \psi(f \chi_{\Delta_n}, P)x$ holds for every hermitian projection $P \in W_p$, every bounding sequence $(\Delta_n)_n$ of f with respect to the spectral measure $M_{\text{Id}_{\mathcal{H}}}$ and every $x \in \mathcal{D}_0$. Using Lemma 5.5 it is also true that for every hermitian projection $P \in W_p$ we have

$$\|\psi(f, P)x - \psi(f \chi_{\Delta_n}, P)x\| \leq \|\psi(f, \text{Id}_{\mathcal{H}})x - \psi(f \chi_{\Delta_n}, \text{Id}_{\mathcal{H}})x\|,$$

and hence, by an analogous estimate as in the proof of Proposition 5.4,

$$(4) \quad \|\psi(f, S_\ell(A))x - \psi(f \chi_{\Delta_n}, S_\ell(A))x\| \leq \|A\| \|\psi(f, \text{Id}_{\mathcal{H}})x - \psi(f \chi_{\Delta_n}, \text{Id}_{\mathcal{H}})x\|.$$

For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have

$$\|\psi(f, \text{Id}_{\mathcal{H}})x - \psi(f \chi_{\Delta_n}, \text{Id}_{\mathcal{H}})x\| \leq \frac{\epsilon}{3\|A\|}.$$

There also exists $L_1(n) \in \mathbb{N}$, such that for every $\ell \geq L_1(n)$ we have

$$\|\psi(f, A)x - \psi(f, S_\ell(A))x\| \leq \frac{\epsilon}{3}.$$

For every $n \in \mathbb{N}$ there also exists $L_2(n) \in \mathbb{N}$, such that for every $\ell \geq L_2(n)$

$$\|\psi(f\chi_{\Delta_n}, A)x - \psi(f\chi_{\Delta_n}, S_\ell(A))x\| \leq \frac{\epsilon}{3}.$$

Let $L(n) = \max\{L_1(n), L_2(n)\}$. Hence, for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$, such that for every $n \geq N_\epsilon$ we have

$$\begin{aligned} \|\psi(f, A)x - \psi(f\chi_{\Delta_n}, A)x\| &\leq \|\psi(f, A)x - \psi(f, S_{L(n)}(A))x\| \\ &\quad + \|\psi(f, S_{L(n)}(A))x - \psi(f\chi_{\Delta_n}, S_{L(n)}(A))x\| \\ &\quad + \|\psi(f\chi_{\Delta_n}, S_{L(n)}(A))x - \psi(f\chi_{\Delta_n}, A)x\| \\ &\leq \epsilon. \end{aligned}$$

Hence, $\psi(f, A)x = \lim_{n \rightarrow \infty} \psi(f\chi_{\Delta_n}, A)x$.

For the other part,

$$\begin{aligned} \psi(f\chi_{\Delta_n}, A)x &= \lim_{\ell \rightarrow \infty} \psi(f\chi_{\Delta_n}, S_\ell(A))x = \lim_{\ell \rightarrow \infty} \psi(f, S_\ell(A))M(\Delta_n)(\text{Id}_{\mathcal{H}})x \\ &= \psi(f, A)M(\Delta_n)(\text{Id}_{\mathcal{H}})x. \quad \square \end{aligned}$$

Proof of Proposition 5.6. We will first prove the linearity in the first factor. For $f, g \in \mathcal{I}(M)$, $\lambda, \mu \in \mathbb{C}$ and $A \in W$, we have to show that $\psi(\lambda f + \mu g, A) = \lambda\psi(f, A) + \mu\psi(g, A)$. We may assume A is positive. Let $S_\ell(A)$ be a limiting sequence of A . For $x \in \mathcal{D}_0$ it is true that

$$\begin{aligned} \psi(\lambda f + \mu g, A)x &= \lim_{\ell \rightarrow \infty} \psi(\lambda f + \mu g, S_\ell(A))x = \lim_{\ell \rightarrow \infty} \sum_{k=1}^{n_\ell} \zeta_{k,\ell} \psi(\lambda f + \mu g, P_{k,\ell})x \\ &= \lim_{\ell \rightarrow \infty} \sum_{k=1}^{n_\ell} \zeta_{k,\ell} (\lambda \psi(f, P_{k,\ell}) + \mu \psi(g, P_{k,\ell}))x \\ &= \lambda \lim_{\ell \rightarrow \infty} \psi(f, \sum_{k=1}^{n_\ell} \zeta_{k,\ell} P_{k,\ell})x + \mu \lim_{\ell \rightarrow \infty} \psi(g, \sum_{k=1}^{n_\ell} \zeta_{k,\ell} P_{k,\ell})x \\ &= \lambda \psi(f, A)x + \mu \psi(g, A)x, \end{aligned}$$

where in the third equality we used the linearity of the integration with respect to the spectral measure $M_{P_{k,\ell}}$ (see Theorem 2.6.(2)).

Now we will prove the linearity in the second factor. We may assume $f \in \mathcal{I}(M)$, $A, B \in W_+$ and $x \in \mathcal{D}_0$. Let Δ_n be a bounding sequence of f with respect to the spectral measure $M_{\text{Id}_{\mathcal{H}}}$. We have $\psi(f, A + B)x = \lim_{n \rightarrow \infty} \psi(f\chi_{\Delta_n}, A + B)x = \lim_{n \rightarrow \infty} (\psi(f\chi_{\Delta_n}, A) + \psi(f\chi_{\Delta_n}, B))x = \psi(f, A)x + \psi(f, B)x$, where in the first equality we used Lemma 5.7 and in the second equality we used the linearity of integration of bounded functions with respect to the non-negative spectral measures (see [16, Proposition 3.5.(3.1)]). \square

Proposition 5.8. *The map \mathbb{I}_M is well-defined.*

To proof Proposition 5.8 we need the following lemma.

Lemma 5.9. *For $f \in \mathcal{I}(M)$ and $A \in W$, we have $\mathcal{D}_0 \subseteq \mathcal{D}(\overline{\psi}(f \otimes A)^*)$ and*

$$\overline{\psi}(f \otimes A)^*x = \overline{\psi}(\overline{f} \otimes A^*)x$$

for every $x \in \mathcal{D}_0$.

Proof. By the decomposition of A into the linear combination of four positive parts and since the domain \mathcal{D}_0 of $\mathcal{D}(\overline{\psi}(f \otimes A))$ is dense in \mathcal{K} , we may assume, by [15, Proposition 1.6(vi)], that A is a positive operator. For $x, y \in \mathcal{D}_0$ and a limiting sequence $S_\ell(A)$ of A , it is true that

$$\begin{aligned} \langle \overline{\psi}(f \otimes A)x, y \rangle &= \langle \lim_{\ell \rightarrow \infty} \overline{\psi}(f \otimes S_\ell(A))x, y \rangle = \lim_{\ell \rightarrow \infty} \langle \overline{\psi}(f \otimes S_\ell(A))x, y \rangle \\ &= \lim_{\ell \rightarrow \infty} \langle x, \overline{\psi}(f \otimes S_\ell(A))^* y \rangle = \lim_{\ell \rightarrow \infty} \langle x, \overline{\psi}(\overline{f} \otimes S_\ell(A))y \rangle = \langle x, \overline{\psi}(\overline{f} \otimes A)y \rangle, \end{aligned}$$

where we used Theorem 2.6.(i) and [15, Proposition 1.6(vi)] in the fourth equality (Since the domain $\mathcal{D}(\overline{\psi}(f \otimes S_\ell(A)))$ is dense,

$$\overline{\psi}(f \otimes S_\ell(A))^* \supseteq \sum_{i=k}^{n_\ell} \zeta_{k,i} \overline{\psi}(f \otimes P_{k,i})^* \supseteq \sum_{i=k}^{n_\ell} \zeta_{k,i} \overline{\psi}(\overline{f} \otimes P_{k,i}) = \overline{\psi}(\overline{f} \otimes S_\ell(A)).$$

Therefore $y \in \mathcal{D}(\overline{\psi}(f \otimes A)^*)$ and $\overline{\psi}(f \otimes A)^* y = \overline{\psi}(\overline{f} \otimes A)y$. \square

Proof of Proposition 5.8. For \mathbb{I}_M to be well-defined, $\overline{\psi}(\sum_{i=1}^n f_i \otimes A_i)$ must be closable for every functions $f_1, \dots, f_n \in \mathcal{I}(M)$ and every $A_1, \dots, A_n \in W$. By [15, Theorem 1.8(i)], it suffices to show that the domain $\mathcal{D}((\overline{\psi}(\sum_{i=1}^n f_i \otimes A_i))^*)$ is dense in \mathcal{K} . Since the domain $\mathcal{D}(\overline{\psi}(\sum_{i=1}^n f_i \otimes A_i))$ is dense, by [15, Proposition 1.6(vi)], $\overline{\psi}(\sum_{i=1}^n f_i \otimes A_i)^* \supseteq \sum_{i=1}^n \overline{\psi}(f_i \otimes A_i)^*$. Therefore it suffices to show that $\sum_{i=1}^n \overline{\psi}(f_i \otimes A_i)^*$ is densely defined. Furthermore, it suffices to prove that every operator $\overline{\psi}(f \otimes A)^*$ with $f \in \mathcal{I}(M)$ and $A \in B(\mathcal{H})$, is defined on \mathcal{D}_0 . But this is the statement of Lemma 5.9, which concludes the proof. \square

5.1. Algebraic properties of \mathbb{I}_M . The main algebraic properties of the integral \mathbb{I}_M are collected in the following theorem.

Theorem 5.10. *For $F, G \in \mathcal{I}(M) \otimes W$, $\alpha, \beta \in \mathbb{C}$, $f, g \in \mathcal{I}(M)$ and a hermitian projection $P \in W_p$, we have:*

- (1) $\mathbb{I}_M(F^*) \subseteq \mathbb{I}_M(F)^*$,
- (2) $\mathbb{I}_M(\alpha F + \beta G) = \overline{\alpha \mathbb{I}_M(F) + \beta \mathbb{I}_M(G)}$,
- (3) $\mathbb{I}_M(FG) \subseteq \overline{\mathbb{I}_M(F) \mathbb{I}_M(G)}$,

To prove Theorem 5.10 we need some additional lemmas.

Lemma 5.11. *For every map $F \in \mathcal{I}(M) \otimes W$ and every compact set K we have*

$$M(K)(\text{Id}_{\mathcal{H}})\overline{\psi}(F) \subseteq \overline{\psi}(F)M(K)(\text{Id}_{\mathcal{H}}) = \overline{\psi}(F\chi_K).$$

Proof. By the linearity of $\overline{\psi}$ and the boundedness of $M(K)(\text{Id}_{\mathcal{H}})$, we may assume $F = f \otimes A$, where $f \in \mathcal{I}(M)$, $A \in W_+$. Now the statement of the lemma is true for every $F = f \otimes S_\ell(A)$, where $S_\ell(A)$ is a limiting sequence of A , by Lemma 5.3. Hence, it holds for $f \otimes A$. \square

Lemma 5.12. *For $F, G \in \mathcal{I}(M) \otimes W$ we have*

$$\overline{\psi}(FG) = \overline{\psi}(F)\overline{\psi}(G).$$

Proof. By the linearity of $\overline{\psi}$, we may assume $F = f \otimes A$, $G = g \otimes B$, where $f, g \in \mathcal{I}(M)$ and $A, B \in W$. Let $y \in \mathcal{D}_0$. Then there is a compact set K and

$x \in \mathcal{K}$, such that $y = M(K)(\text{Id}_{\mathcal{H}})x$. Let $\Delta_1 := K$ and Δ_m be a common bounding sequence for fg , f and g . For a fixed $n \in \mathbb{N}$, we have

$$\begin{aligned}
 & \overline{\psi}(fg \otimes AB)y = \\
 &= \lim_{m \rightarrow \infty} \overline{\psi}(fg\chi_{\Delta_m} \otimes AB)y \\
 &= \lim_{m \rightarrow \infty} \overline{\psi}(f\chi_{\Delta_m} \otimes A) \circ \overline{\psi}(g\chi_{\Delta_m} \otimes B)y \\
 &= \lim_{m \rightarrow \infty} \overline{\psi}(f\chi_{\Delta_m} \otimes A) \circ \overline{\psi}(g\chi_{\Delta_m} \otimes B) \circ M(\Delta_n)(\text{Id}_{\mathcal{H}}) \circ M(K)(\text{Id}_{\mathcal{H}})x \\
 &= \lim_{m \rightarrow \infty} \overline{\psi}(f\chi_{\Delta_n} \otimes A) \circ \overline{\psi}(g\chi_{\Delta_m}\chi_{\Delta_n} \otimes A) \circ M(K)(\text{Id}_{\mathcal{H}})x \\
 &= \overline{\psi}(f \otimes A) \circ \overline{\psi}(g\chi_{\Delta_n} \otimes B) \circ M(K)(\text{Id}_{\mathcal{H}})x
 \end{aligned}$$

where we used Lemma 5.7 in the first and the fifth equality, [16, Proposition 7.2.] in the second equality together with the fact that $\text{Im}(\overline{\psi}) \subset \mathcal{D}_0$ (Indeed,

$$\begin{aligned}
 M(K)(\text{Id}_{\mathcal{H}})\overline{\psi}(f \otimes A)M(K)(\text{Id}_{\mathcal{H}})x &= \overline{\psi}(f \otimes A)M(K)(\text{Id}_{\mathcal{H}})^2x \\
 &= \overline{\psi}(f \otimes A)M(K)(\text{Id}_{\mathcal{H}})x,
 \end{aligned}$$

the fact that M is a non-negative spectral measure in the third inequality (and $\Delta_n \supseteq \Delta_1$) and the equality part of Lemma 5.7 in the fourth equality. As $n \rightarrow \infty$, we get, by the use of Lemma 5.7, $\overline{\psi}(fg \otimes AB)y = \overline{\psi}(f \otimes A)\overline{\psi}(g \otimes B)y$. This concludes the proof. \square

Proof of Theorem 5.10. For $x \in \mathcal{D}_0$ and $F := \sum_{i=1}^n f_i \otimes A_i$, where $f_1, \dots, f_n \in \mathcal{I}(M)$, $A_1, \dots, A_n \in W$, it is true that

$$\begin{aligned}
 \mathbb{I}_M(F^*)x &= \overline{\psi}(F^*)x = \sum_{i=1}^n \overline{\psi}(\overline{f_i} \otimes A_i^*)x = \sum_{i=1}^n \overline{\psi}(f_i \otimes A_i)^*x \\
 &= \left(\sum_{i=1}^n \overline{\psi}(f_i \otimes A_i) \right)^*x = \overline{\psi}(F)^*x = \mathbb{I}_M(F)^*x,
 \end{aligned}$$

where the first and the sixth equality follow by the definition of $\overline{\psi}$ and \mathbb{I}_M , the second and the fifth by the linearity of $\overline{\psi}$, the third equality follows by Lemma 5.9 and in the forth equality we used [15, Proposition 1.6(vi)] ($\mathcal{D}_0 = \mathcal{D}(\sum_{i=1}^n \overline{\psi}(f_i \otimes A_i)$ is dense in \mathcal{K}). Since $\mathbb{I}_M(F)^*$ is the closed extension of the operator $\overline{\psi}(F^*)$, and $\mathbb{I}_M(F^*)$ is its closure, part (1) is true.

Now we prove $\mathbb{I}_M(F + G) = \overline{\mathbb{I}_M(F) + \mathbb{I}_M(G)}$. We have

$$\begin{aligned}
 \mathbb{I}_M(F + G) &= \overline{\overline{\psi}(F + G)} = \overline{\overline{\psi}(F + G)^{**}} \supseteq \overline{(\overline{\psi}(F)^* + \overline{\psi}(G)^*)^*} \\
 &\supseteq \overline{\overline{\psi}(F)^{**} + \overline{\psi}(G)^{**}} = \overline{\overline{\psi}(F)} + \overline{\overline{\psi}(G)} = \mathbb{I}_M(F) + \mathbb{I}_M(G) \supseteq \overline{\psi}(F) + \overline{\psi}(G),
 \end{aligned}$$

where the first \supseteq follows by $\mathcal{D}_0 \subseteq \mathcal{D}(\overline{\psi}(F)^*)$ for every $F \in \mathcal{B} \otimes W$, by Lemma 5.9. Hence, $\mathbb{I}_M(F) + \mathbb{I}_M(G)$ is closable and the equality $\mathbb{I}_M(F + G) = \overline{\mathbb{I}_M(F) + \mathbb{I}_M(G)}$ is true.

By Lemma 5.12, \mathcal{D}_0 is contained in the domain of the operator $\mathbb{I}_M(G^*)\mathbb{I}_M(F^*)$. Since $(\mathbb{I}_M(F)\mathbb{I}_M(G))^* \supseteq \mathbb{I}_M(G)^*\mathbb{I}_M(F)^*$ (by [15, Proposition 1.7(i)]) and also $\mathbb{I}_M(G)^*\mathbb{I}_M(F)^* \supseteq \mathbb{I}_M(G^*)\mathbb{I}_M(F^*)$ by part (1), the operator $\mathbb{I}_M(F)\mathbb{I}_M(G)$ is closable (by [15, Proposition 1.8(i)]). For $x \in \mathcal{D}_0$, $\mathbb{I}_M(FG)x = \mathbb{I}_M(F)\mathbb{I}_M(G)x$ (by Lemma 5.12). Since $\mathbb{I}_M(FG)$ is the closure for $\overline{\psi}(FG)$ and $\overline{\mathbb{I}_M(F)\mathbb{I}_M(G)}$ is the closed extension for $\overline{\psi}(FG)$, part (3) follows. \square

6. PROOF OF THEOREM D

Assume the notation as in Theorem D. In the proof we will need the following observation.

Proposition 6.1. *For every $k \in \mathcal{K}$, such that for all $b \in \mathcal{B}$ the maps*

$$(\overline{\rho_b})_k : W \rightarrow \mathbb{C}, \quad (\overline{\rho_b})_k(A) := \left\langle \overline{\rho(b \otimes A)} k, k \right\rangle$$

are well-defined, the map $(\overline{\rho_b})_k$ is a bounded linear functional.

Proof. Choose $k \in \mathcal{K}$, such that for all $b \in \mathcal{B}$ the maps $(\overline{\rho_b})_k$ are well-defined. Fix $b \in \mathcal{B}$. First we argue, that $(\overline{\rho_b})_k$ is a linear functional. The homogeneity of $(\overline{\rho_b})_k$ is clear. For additivity we notice that

$$\begin{aligned} \overline{\rho(b \otimes (A + B))} &= \overline{\rho(b \otimes A) + \rho(b \otimes B)} = (\rho(b \otimes A) + \rho(b \otimes B))^{**} \\ &\supseteq (\rho(b \otimes A)^* + \rho(b \otimes B)^*)^* \supseteq \rho(b \otimes A)^{**} + \rho(b \otimes B)^{**} = \overline{\rho(b \otimes A)} + \overline{\rho(b \otimes B)}, \end{aligned}$$

where we used [15, Theorem 1.8(ii)] for the second and the third equality and [15, Proposition 1.6(vi)] for both inclusions \supseteq .

The proof of boundedness of $(\overline{\rho_b})_k$ is the same to the proof of boundedness of $(\rho_b)_k$ in Proposition 3.1 above. \square

We also need the following easy observation.

Proposition 6.2. *For every $b \in \mathcal{B}$ we have $\mathcal{D}(\overline{\rho(b \otimes \text{Id}_{\mathcal{H}})}) \subseteq \mathcal{D}(\overline{\rho(b \otimes A)})$ for every $A \in W$. In particular, $\cup_{K \in \mathcal{K}} \mathcal{D}_{\alpha_K, \rho|_{\text{Id}_{\mathcal{H}}}} \subseteq \mathcal{D}(\overline{\rho(b \otimes A)})$ for every $b \in \mathcal{B}$ and every $A \in W$.*

Proof. First let us prove, that for every $b \in \mathcal{B}$ and every hermitian projection $P \in W_p$ the inclusion $\mathcal{D}(\overline{\rho(b \otimes \text{Id}_{\mathcal{H}})}) \subseteq \mathcal{D}(\overline{\rho(b \otimes P)})$ of domains is true. By the linearity of ρ , $\rho(b \otimes \text{Id}_{\mathcal{H}}) = \rho(b \otimes P) + \rho(b \otimes (\text{Id}_{\mathcal{H}} - P))$. By ρ being a $*$ -representation on \mathcal{D} , for every $k_1, k_2 \in \mathcal{D}$ the following is true

$$\begin{aligned} \langle \rho(b \otimes P)k_1, \rho(b \otimes (\text{Id}_{\mathcal{H}} - P))k_2 \rangle &= \langle \rho(b \otimes (\text{Id}_{\mathcal{H}} - P))^* \rho(b \otimes P)k_1, k_2 \rangle \\ &= \langle \rho(b^*b \otimes (\text{Id}_{\mathcal{H}} - P)P)k_1, k_2 \rangle = \langle \rho(0)k_1, k_2 \rangle = 0. \end{aligned}$$

Hence, $\rho(b \otimes P)\mathcal{D} \perp \rho(b \otimes (\text{Id}_{\mathcal{H}} - P))\mathcal{D}$. Thus, the inclusion $\mathcal{D}(\overline{\rho(b \otimes \text{Id}_{\mathcal{H}})}) \subseteq \mathcal{D}(\overline{\rho(b \otimes P)})$ holds.

Now we will prove the inclusion $\mathcal{D}(\overline{\rho(b \otimes \text{Id}_{\mathcal{H}})}) \subseteq \mathcal{D}(\overline{\rho(b \otimes A)})$ of domains for an arbitrary operator $A \in W$. Let $k \in \mathcal{D}(\overline{\rho(b \otimes \text{Id}_{\mathcal{H}})})$. Let k_i be the sequence from \mathcal{D} , such that $k_i \rightarrow k$ and $\rho(b \otimes \text{Id}_{\mathcal{H}})k_i \rightarrow \rho(b \otimes \text{Id}_{\mathcal{H}})k$. For the inclusion $k \in \mathcal{D}(\overline{\rho(b \otimes A)})$, the sequence $\rho(b \otimes A)k_i$ has to be convergent. We have

$$\|\rho(b \otimes A)k_i - \rho(b \otimes A)k_j\| = \langle \rho(b^*b \otimes A^*A)(k_i - k_j), k_i - k_j \rangle$$

Let $S_\ell(A^*A) = \sum_{k=1}^{n_\ell} \zeta_{k,\ell} R_{k,\ell}$ be a limiting sequence of A^*A . It can be chosen to be increasing. Then also the sequence $\langle \rho(b^*b \otimes S_\ell(A^*A))(k_i - k_j), k_i - k_j \rangle$ is increasing and converges to $\langle \rho(b^*b \otimes A^*A)(k_i - k_j), k_i - k_j \rangle$, by Proposition 6.1. For every $\epsilon > 0$ we can choose $N \in \mathbb{N}$, such that for $i, j > N$ $\langle \rho(b^*b \otimes \text{Id}_{\mathcal{H}})(k_i -$

$k_j), k_i - k_j \rangle < \frac{\epsilon}{\|A^*A\|}$. Hence, for every $\ell \in \mathbb{N}$ we have

$$\begin{aligned} \langle \rho(b^*b \otimes S_\ell(A^*A))(k_i - k_j), k_i - k_j \rangle &= \sum_{k=1}^{n_\ell} \zeta_{k,\ell} \langle \rho(b^*b \otimes R_{k,\ell})(k_i - k_j), k_i - k_j \rangle \\ &\leq \|A^*A\| \langle \rho(b^*b \otimes \text{Id}_{\mathcal{H}})(k_i - k_j), k_i - k_j \rangle \\ &\leq \epsilon. \end{aligned}$$

Hence, $\rho(b \otimes A)k_i$ is convergent and so $k \in \mathcal{D}(\overline{\rho(b \otimes A)})$. \square

Proof of Theorem D. Direction (2) \Rightarrow (1) follows by Propositions 1.3 and 1.4.

Direction (1) \Rightarrow (2). By the assumption, the maps ρ_P are integrable *-representations for every $P \in W_p$. By Theorem C', there exist unique regular spectral measures $E_P : \text{Bor}(\widehat{B}) \rightarrow B(\mathcal{K})$ such that $\overline{\rho_P(b)}x = \int_{\widehat{B}} f_b(\chi) dE_P(\chi)$ holds for every $x \in \mathcal{D}(\overline{\rho_P(b)})$, $b \in \mathcal{B}$, $P \in W_p$ and the support $\text{supp}((E_{\text{Id}_{\mathcal{H}}})_{x,x})$ is compact for every $k \in \cup_{K \in \mathcal{K}} \mathcal{D}_{\alpha_K, \rho_{\text{Id}_{\mathcal{H}}}}$. The idea is to show that the family $\{E_P\}_{P \in W_p}$ satisfies the conditions of Theorem 2.2 to obtain a non-negative spectral measure M representing ρ .

By an analogous proof as in the proof of Theorem C', we prove the containment $\text{supp}((E_P)_{k,k}) \subseteq \text{supp}((E_{\text{Id}_{\mathcal{H}}})_{k,k})$ for every $P \in W_p$ and every $k \in \cup_{K \in \mathcal{K}} \mathcal{D}_{\alpha_K, \rho_{\text{Id}_{\mathcal{H}}}}$.

It remains to check first that the family $\{E_P\}_{P \in W_p}$ satisfies the conditions of Theorem 2.2, second that M is a representing measure of $\overline{\rho}$ and finally that M is unique, regular and normalized. Since the support of every E_P is contained in the compact set $K_k = \text{supp}((E_{\text{Id}_{\mathcal{H}}})_{k,k})$ for every $k \in \cup_{K \in \mathcal{K}} \mathcal{D}_{\alpha_K, \rho_{\text{Id}_{\mathcal{H}}}}$, the proofs of the conditions of Theorem 2.2 are the same as the proofs of the corresponding conditions of [16, Theorem 9.1], just that we replace the use of [16, Lemma 2.3] by Lemma 3.2 above. By the polarization argument and by the density of $\cup_{K \in \mathcal{K}} \mathcal{D}_{\alpha_K, \rho_{\text{Id}_{\mathcal{H}}}}$, they are true for every $k \in \mathcal{K}$.

M is the representing measure of ρ : We have

$$\begin{aligned} \overline{\rho(\sum_{i=1}^n b_i \otimes A_i)} &= \overline{\rho(\sum_{i=1}^n b_i \otimes A_i)^{**}} = \overline{(\sum_{i=1}^n \rho(b_i \otimes A_i))^{**}} \\ &\supseteq \sum_{i=1}^n \overline{\rho(b_i \otimes A_i)^{**}} = \sum_{i=1}^n \overline{\rho(b_i \otimes A_i)}. \end{aligned}$$

Hence, $\overline{\rho(\sum_{i=1}^n b_i \otimes A_i)} = \overline{\sum_{i=1}^n \overline{\rho(b_i \otimes A_i)}}$. Therefore, it suffices to prove that $\overline{\rho(b \otimes A)} = \int_X (f_b \otimes A) dM$. For a hermitian projection $A \in W_p$, this is true by the construction of measures M_A . Let now $A \in W_+$ be a positive operator. By Proposition 6.1, $(\overline{\rho(b)})_k$ is bounded for every $k \in \cup_{K \in \mathcal{K}} \mathcal{D}_{\alpha_K, \rho_{\text{Id}_{\mathcal{H}}}}$. Let $K_0 = \text{supp}((E_{\text{Id}_{\mathcal{H}}})_{k,k})$. Therefore,

$$\begin{aligned} (\overline{\rho(b)})_k(A) &= \langle \overline{\rho(b \otimes A)}k, k \rangle = \lim_{\ell} \langle \overline{\rho(b \otimes S_\ell(A))}k, k \rangle \\ &= \lim_{\ell} \langle (\int_X (f_b \otimes S_\ell(A)) dM) k, k \rangle = \lim_{\ell} \langle (\int_{K_0} (f_b \otimes S_\ell(A)) dM) k, k \rangle \\ &= \langle (\int_{K_0} (f_b \otimes A) dM) k, k \rangle = \langle (\int_X (f_b \otimes A) dM) k, k \rangle \end{aligned}$$

By the polarization argument, $\langle \overline{\rho(b \otimes A)}k_1, k_2 \rangle = \langle (\int_X (f_b \otimes A) dM) k_1, k_2 \rangle$ for every $k_1, k_2 \in \cup_{K \in \mathcal{K}} \mathcal{D}_{\alpha_K, \rho_{\text{Id}_{\mathcal{H}}}}$. By the density of $\cup_{K \in \mathcal{K}} \mathcal{D}_{\alpha_K, \rho_{\text{Id}_{\mathcal{H}}}}$, $\overline{\rho(b \otimes A)} =$

$\int_X (f_b \otimes A) dM$. We decompose an arbitrary A into the usual linear combination of four positive parts and use the result for each of them.

Finally, M is unique, regular and normalized, by the uniqueness and the regularity of each E_P and the unitality of ρ . \square

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